1. (5+5=10 pts) Consider the following 4×4 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{bmatrix}$$

5

(a) What is rank(A)?

5

(b) Is A invertible? Why? If yes, please compute A^{-1}.

1a reduce to rref

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{bmatrix}$$

$3(\mathrm{I}) - \mathrm{II}$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{bmatrix}$$

$\mathrm{II} \div 2$

$7\mathrm{III} - 5\mathrm{IV}$

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\mathrm{I} - 2\mathrm{II}$

$\mathrm{III} - 3\mathrm{IV}$

$\mathrm{IV} \div 2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\mathrm{III} \div 5$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\mathrm{rank}(A) = 4}$

b A is invertible because its rref reduces to an Identity matrix

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 8 & 0 & 0 & 0 & 1 \end{array}\right]$$

$3(\mathrm{I}) - \mathrm{II}$ $7(\mathrm{III}) - 5\mathrm{IV}$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 3 & -1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 7 & -5 \end{array}\right]$$

$(\mathrm{I}) - (\mathrm{II})$ $\mathrm{II} \div 2$

$\mathrm{III} - 3\mathrm{IV}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & -20 & 15 \\ 0 & 0 & 0 & 2 & 0 & 0 & 7 & -5 \end{array}\right]$$

$\mathrm{III} \div 5$ $\mathrm{IV} \div 2$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{7}{2} & -\frac{5}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & \frac{7}{2} & -\frac{5}{2} \end{bmatrix}$$

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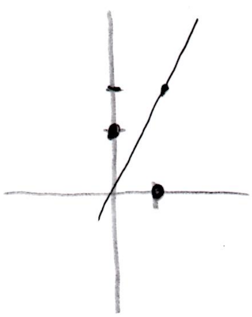
2. (10 pts) Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, which is defined as

$$T(\vec{x}) = \text{ref}_{L_1}(\text{ref}_{L_2}(\vec{x})).$$

Here $\vec{x}$ is any vector in $\mathbb{R}^2$, ref_{L_1} is the reflection about the line $y = 2x$, ref_{L_2} is the reflection about the y-axis. If $\vec{x} = (1, 1)$, please compute $T(\vec{x})$.

(-1, 1)

reflection about $y = 2x$



reflection $= 2\left(\frac{x \cdot w}{w \cdot w}\right)\vec{w} - \vec{x}$

$\vec{x} = (1,1)$

we can find

reflection about y axis

$e_1 \to T(e_1) \to \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\cancel{e_2} \; T(e_2) \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

first reflect over y axis

$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

now reflection =

$2\left(\frac{x \cdot w}{w \cdot w}\right)\vec{w} - \vec{x}$

$\vec{w} = (1, 2)$

$\vec{x} = (-1, 1)$

$2\left(\frac{(-1,1)\cdot(1,2)}{(1,2)\cdot(1,2)}\right)(1,2) - (-1,1)$

$2\left(\frac{-1+2}{1+4}\right)(1,2) - (-1,1)$

$2\left(\frac{1}{5}\right)(1,2) - (-1,1)$

$\left(\frac{2}{5}, \frac{4}{5}\right) - (-1,1)$

$\boxed{T(\vec{x}) = \left(\frac{7}{5}, -\frac{1}{5}\right) \text{ or } \begin{bmatrix} \frac{7}{5} \\ -\frac{1}{5} \end{bmatrix}}$

3. (10 pts) Let $\vec{x}$ be a vector in $\mathbb{R}^2$. It is first reflected on the x-axis then rotated counterclockwise by $\pi/4$. It turns out that it is transformed to be the vector $(1, -1)$. What is the original $\vec{x}$?

rotated counterclockwise by $\pi/4$

inverse is rotate clockwise by $\pi/4$

So transformation matrix

$$\begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{2\times 1}$$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix}$$

reflect over x-axis

is the inverse of reflecting over x-axis

$$e_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to T(e_1) \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to T(e_2) \to \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

thus by applying the inverse to our point

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix}$$

we get

$$\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

$\boxed{\text{The original } \vec{x} \text{ was } (0, \sqrt{2})}$

10

2 4 6 8
2 4 2 8
0 0 4 0

4. (10 pts) Let matrix A be

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 2 & 8 \end{bmatrix}$$

Find vectors that span the kernel of A.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & 2 & 8 & 0 \end{array}\right]$$

$$\boxed{\ker(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

$2\mathrm{I} - \mathrm{II}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 4 & 0 & 0 \end{array}\right]$$

$\mathrm{II} \div 4$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$\mathrm{I} - 3\mathrm{II}$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$x_1 = -2s - 4t$

$x_2 = s$

$x_3 = 0$

$x_4 = t$

$$\begin{bmatrix} -2s-4t \\ s \\ 0 \\ t \end{bmatrix} \rightarrow s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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3 15 27 12
3 7 11 12
0 8 16 0

2 10 18 8
2 6 10 8
0 4 8 0

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5. (10 pts) Is the vector $w = (4, 8, 12)$ in the span of vectors $v_1 = (1, 5, 9)$, $v_2 = (2, 6, 10)$ and $v_3 = (3, 7, 11)$? In other words, do there exist numbers a, b, c such that $w = av_1 + bv_2 + cv_3$?

$$\left[\begin{array}{ccc|c} 1 & 5 & 9 & 4 \\ 2 & 6 & 10 & 8 \\ 3 & 7 & 11 & 12 \end{array}\right]$$

$2(\mathrm{I}) - \mathrm{II} \quad 3(\mathrm{I}) - \mathrm{III}$

$$\left[\begin{array}{ccc|c} 1 & 5 & 9 & 4 \\ 0 & 4 & 8 & 0 \\ 0 & 8 & 16 & 0 \end{array}\right]$$

$2(\mathrm{II}) - (\mathrm{III})$

$$\left[\begin{array}{ccc|c} 1 & 5 & 9 & 4 \\ 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathrm{II} \div 4$

$$\begin{bmatrix} 1 & 5 & 9 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

1 5 9 4
0 5 10 0

$\mathrm{I} - 5\,\mathrm{II}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$c = t$

$b = -2t$

$a = 4 + t$

$$\begin{bmatrix} 4+t \\ -2t \\ t \end{bmatrix}$$

vector w is in the span of vectors v_1, v_2 and v_3 as there are infinitely many solutions of a, b, c such that $w = av_1 + bv_2 + cv_3$

$a = 4 + t \quad b = -2t \quad c = t$

6. (bonus 10 pts) There is a bank account with a balance of x dollars today ($x > 0$). From today, y dollars are deposited into the account everyday unless the balance is zero. Also a fixed amount of money is withdrawn from the account everyday. If z dollars are withdrawn everyday, then the account will have zero balance after 6 days. If $0.8z$ dollars are withdrawn everyday, then the account will sustain exactly 9 days. If $0.6z$ dollars are withdrawn everyday, after how many days the account will be drained?

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$6y = 6z$

$y = z$

$9y = 7.2z$

$y =$

$(1, 1)$

$6 - t \quad y \qquad 6 - t \quad -6 + .6t = 0$

$x + 6y - 6z = 0 \qquad t = \text{the}$

$x + 9y - .8z(9) = 0$

$x + 6y - 6z = 0$

$x + 9y - 7.2z = 0$

$x + ty - .6tz = 0$

$$\left[\begin{array}{ccc|c} 1 & 6 & -6 & 0 \\ 1 & 9 & -7.2 & 0 \\ 1 & t & -.6t & 0 \end{array}\right]$$

I − II III − I

$$\left[\begin{array}{ccc|c} 1 & 6 & -6 & 0 \\ 0 & -3 & 1.2 & 0 \\ 0 & t-6 & -.6t+6 & 0 \end{array}\right]$$

II ÷ 3

$$\left[\begin{array}{ccc|c} 1 & 6 & -6 & 0 \\ 0 & -1 & .4 & 0 \\ 0 & t-6 & -.6t+6 & 0 \end{array}\right]$$

II ÷ −1

$6 - t = -3$

$-t = -9$

$t = 9$

$6y = 6z$

$y = z$

$6 - t =$

$$\left[\begin{array}{l} 6y - 6z = -x \\ 9y - 7.2z = -x \end{array}\right]$$

6 (I) − 4 (II)

$$\left[\begin{array}{cc} 6 & -6 = -x \\ 0 & -1.2 = -2x \end{array}\right]$$

$z =$

$$\left[\begin{array}{ccc|c} 1 & 6 & -6 & 0 \\ 0 & 1 & -.4 & 0 \\ 0 & t-6 & -.6t+6 & 0 \end{array}\right]$$

III − (t−6)II

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~~There appears to be a linear relationship between reduction of z to increase in sustainability to account when z decreases by .2 the days increase by 3. If .6z dollars are withdrawn it appears that the account will be drained after~~ $\boxed{12 \text{ days}}$

$$\left[\begin{array}{ccc|c} 1 & 6 & -6 & 0 \\ 0 & 1 & -0.4 & 0 \\ 0 & 0 & 0.4(t-6)-0.6t+6 & 0 \end{array}\right]$$

Look at just row 3

$0.4t - 2.4 - 0.6t + 6 = 0$

$-0.2t + 3.6 = 0$

$0.2t = 3.6$

Real Answer ⇒ $\boxed{t = 18}$