M33A Midterm 2: Version B

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Section: 2E

DO NOT OPEN THIS EXAM UNTIL TOLD TO DO SO.
JUSTIFY ALL ANSWERS UNLESS EXPLICITLY TOLD OTHERWISE.

Question	Points	Score
1	3	2
2	4	4
3	5	5
4	4	4
5	4	~~2~~ 3 MG
6	5	3
Total:	25	21

1. (a) (1 point) The *nullity* of a matrix A is the dimension of $\underline{\quad \ker(A) \quad}$.

   (b) (1 point) The *rank* of a matrix A is the dimension of $\underline{\quad im(A) \quad}$.

   (c) (1 point) State the Rank-Nullity theorem for an $m \times n$ matrix A.

$$m = rank(A) + nullity(A)$$
$$= \dim(im(A)) + \dim(\ker(A))$$

2. True or false. (No justification necessary.)

   (a) (1 point) Every subspace of $\mathbb{R}^5$ can be spanned by 7 vectors.

TRUE

   (b) (1 point) For any nonzero vector $\vec{v} \in \mathbb{R}^n$, there is a *unique* unit vector $\vec{u} \in \mathbb{R}^n$ and a *unique* real number $a \in \mathbb{R}$ such that $\vec{v} = a\vec{u}$.

FALSE

$\ker(AB) = \{\vec{0}\}$ $\quad$ $\ker(A) = \{\vec{0}\}$

4

   (c) (1 point) If the product AB of two matrices is one-to-one, then A is one-to-one.

FALSE

   (d) (1 point) If the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \in \mathbb{R}^9$ are linearly independent, then none of them is a scalar multiple of any other.

TRUE

3. (5 points) Find the QR factorization of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & -3 \\ 1 & -3 \end{bmatrix}$$

$\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \\ -3 \end{bmatrix}$

$\vec{v}_1 = \sqrt{2}\, u_1$

$u_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -3 \\ -3 \end{bmatrix} = -\dfrac{3}{\sqrt{2}} - \dfrac{3}{\sqrt{2}} = -\dfrac{6}{\sqrt{2}}$

$\vec{v}_2^{\perp} = \vec{v}_2 - (u_1 \cdot \vec{v}_2) u_1$

$= \begin{bmatrix} 1 \\ 1 \\ -3 \\ -3 \end{bmatrix} - \left(-\dfrac{6}{\sqrt{2}}\right) \begin{bmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$\dfrac{6}{\sqrt{2}} \cdot \dfrac{1}{\sqrt{2}} = \dfrac{6}{2} = 3$

$= \begin{bmatrix} 1 \\ 1 \\ -3 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 1 & -3 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & -6/\sqrt{2} \\ 0 & \sqrt{2} \end{bmatrix}$

$Q \qquad R$

$\therefore u_2 = \dfrac{\vec{v}_2^{\perp}}{\|v_2^{\perp}\|} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$

5

$\sqrt{2}\, u_2 = \vec{v}_2^{\perp}$

$\downarrow$

$\vec{v}_2 = \sqrt{2}\, u_2 - \dfrac{6}{\sqrt{2}} u_1$

4. (4 points) Find a basis for the kernel of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 2 & 1 \end{bmatrix} \begin{matrix} \\ -(I) \\ -(I) \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 \end{bmatrix} \begin{matrix} \\ \\ -2(II) \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{ccccc} & & & & \\ \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & & \uparrow & \\ x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix}$$

basis for the im (A) $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$x_1 = 0$

$x_2 = -x_3$

$x_4 = -\frac{1}{2} x_5$

let $x_3 = t$, $x_5 = S$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ -t \\ t \\ -1/2 S \\ S \end{bmatrix}$$

$$= t \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1/2 \\ 1 \end{bmatrix}$$

basis for the ker (A) $= \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right\}$

5. (4 points) Let $\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$. Find $[A]_\beta$.

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$S^{-1} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] + (\mathrm{II})$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{array}\right] - (\mathrm{III})$$

$-\frac{1}{2} + \frac{3}{2}$

$-\frac{1}{2} - \frac{3}{2}$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 \end{array}\right]$$

$\frac{3}{2} + \frac{1}{2}$

$\frac{3}{2} - \frac{1}{2} =$

$\therefore \quad A = S[A]_\beta$

$$[A]_\beta = S^{-1}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

3

3×2

6. Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & -1 \end{bmatrix}$. Notice that the matrix equation $A\vec{x} = \vec{b}$ does not always have a solution, since A is not surjective.

(a) (2 points) Find an orthonormal basis $\alpha = \{\vec{u}_1, \vec{u}_2\}$ for the image of A.

2

basis of the im(A) = $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$,

$\vec{v}_2^{\perp} = \vec{v}_2 - (v_2 \cdot u_1) u_1$

$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 0 u_1$

$= \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\therefore \vec{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

$\therefore$ basis $\alpha = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} \right\}$

1 (b) (1 point) Find the 3×3 matrix M that represents the orthogonal projection $P_{\mathrm{im}(A)}$ onto the image of A.

$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$

$M = QQ^T$

$= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ -1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$

$= \begin{bmatrix} 5/6 & -1/3 & -1/6 \\ -1/3 & 1/3 & -1/3 \\ -1/6 & -1/3 & 5/6 \end{bmatrix}$

$M\vec{x} = \vec{b}$

0 (c) (2 points) Write $\vec{b} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$ as a sum of a vector in $\mathrm{im}(A)$ and a vector in $\mathrm{im}(A)^{\perp}$.

$M\vec{x} = \vec{b}$

$\mathrm{im}(A)^{\perp} = \ker Q^T$

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$\downarrow$

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$x_1 = -x_3$

$x_2 = 0$

let $x_3 = t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

basis of the $\ker Q^T$

$= \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$