M33A Midterm 2: Version B

November 20, 2017

DO NOT OPEN THIS EXAM UNTIL TOLD TO DO SO.
JUSTIFY ALL ANSWERS UNLESS EXPLICITLY TOLD OTHERWISE.

Question	Points	Score
1	20	20
2	10	10
3	30	30
4	25	23
5	15	20
Total:	100	103

Wow!

1. True or false, no justification required.

(a) (5 points) For any matrix A, $\ker(A)^{\perp} = \text{im}(A^{T})$.

5 True

(b) (5 points) For any 6×4 matrix A, the equation $A\vec{x} = \vec{b}$ has a *unique* least-squares solution.

5 False

(c) (5 points) If the rows of an $n \times n$ matrix A form a basis of $\mathbb{R}^n$, then $\det(A) = 1$.

5 False

(d) (5 points) If A is a 3×3 matrix and $\lambda \in \mathbb{R}$, then $\det(\lambda A) = \lambda^3 \det(A)$.

5 True

2. (a) (5 points) Give an example of a 2×2 skew-symmetric orthogonal matrix.

5 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(b) (5 points) There are no 3×3 skew-symmetric orthogonal matrices. Why? *(Hint: Use determinants.)*

3x3 skew symmetric matrices are of the form

5 $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$, so $\det(A) = -abc + abc = 0$

Since $\det(A) = 0$, A is not invertible so it can[not be] orthogonal

3. Let $V \subseteq \mathbb{R}^3$ be the subspace with basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(a) (15 points) Find an orthonormal basis $\mathcal{U}$ for V via the Gram–Schmidt process, starting with $\mathcal{B}$.

$$V_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \qquad u_1 = \frac{V_1}{\|V_1\|} = \frac{1}{\sqrt{1+4+4}} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad u_2 = \frac{V_2 - (V_2 \cdot u_1)u_1}{\|V_2 - (V_2 \cdot u_1)u_1\|} = \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \right) \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}}{\| \quad " \quad \|}$$

$$= \frac{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}}{\| \quad " \quad \|} = \frac{\begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}}{\left\| \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \right\|} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$\mathcal{U} = \left\{ \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} \right\}$$

15

(continued on next page)

4. (a) (5 points) If Q is a $m \times n$ matrix with orthonormal columns, what linear transformation does Q^TQ represent? (No justification required.)

5 $\quad n \times m \quad m \times n$

$$Q^TQ = I_n$$

(b) (5 points) If Q is a $m \times n$ matrix with orthonormal columns, what linear transformation does QQ^T represent? (No justification required.)

3 $\quad$ Orthogonal projection on ?

(c) (15 points) Find the QR factorization of the matrix

15

$$M = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 1 & 0 \\ 0 & 4 & -3 \end{bmatrix}.$$

$$Q = \begin{bmatrix} 0 & 3/5 & 4/5 \\ 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$u_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad R_{11} = \|v_1\| = 1$

$$u_2 = \frac{v_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1}{\|v_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1\|} = \frac{\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - \left(\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\| \quad " \quad \|} = \frac{\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\| \quad " \quad \|} = \frac{\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}}{\left\|\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}\right\|} = \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}$$

$R_{22} = \|V_2 - (\vec{V}_2 \cdot \vec{u}_1)u_1\| = 5$

$$u_3 = \frac{V_3 - (\vec{V}_3 \cdot \vec{U}_1)\vec{U}_1 - (\vec{V}_3 \cdot U_2)V_2}{\| \quad " \quad \|}$$

$R_{12} = u_1 \cdot V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = 1$

$$= \frac{\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \left[\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix}\right]u_2}{\| \quad " \quad \|} = \frac{\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}}{\left\|\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}\right\|} = \begin{bmatrix} 4/5 \\ 0 \\ -3/5 \end{bmatrix}$$

$R_{13} = u_1 \cdot V_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = 0$

$R_{23} = U_2 \cdot V_3 = \begin{bmatrix} 3/5 \\ 0 \\ 4/5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix} = 0$

$R_{33} = \left\|\begin{matrix} 4 \\ 0 \\ -3 \end{matrix}\right\| = 5$

5. Let $A = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$.

(a) (5 points) Compute $\det(A)$. Is A invertible?

$\det(A) = ad - bc = -2 - 4 = -6$

5

yes it is invertible

(b) (10 points) Use the determinant to find all λ such that the matrix $A - \lambda I_2$ is not invertible. (Such a λ is called an *eigenvalue* of the matrix A.)

$\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}$

$\det(A - \lambda I_2) = (2-\lambda)(-1-\lambda) - 4 = \lambda^2 - \lambda - 2 - 4$

10

$= \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2)$

$\lambda = -2, 3$

(c) (5 points (bonus)) Is A similar to a diagonal matrix? If so, which one? If not, why?

$\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

+5