Math 33A - Lectures 3 and 4
Fall 2018

Midterm 2

Instructions: You have 60 minutes to complete this exam. There are five questions, worth a total of 50 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Unless instructed otherwise, you need to justify your answers. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use the scrap pages at the end of this booklet.

Please take a moment to ensure that your booklet consists of ten pages, the last three being reserved for additional work.

Do not forget to write your full name, section and UID in the space below. For identification purposes, please sign below.

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Question	Points	Score
1	8	4
2	10	10
3	11	11
4	9	9
5	12	11
Total:	50	45

Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

on diag

(a) [4pts.] A is a 3×3 matrix with $A^T A = -I_3$.

Not possible because $A^T A$ is always positive ~~equal to Identity matrix, so $A^T A = +I_3$~~

~~if it is square~~

$A^T A = -I_3 = -\begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}$

~~$A^T = A^{-1}$ in general b/c $A^T A = I_3$ in general~~

if $A = \begin{bmatrix} a&b&c\\d&e&f\\g&h&i \end{bmatrix}$, then calculate $A^T A$

$A^T A = \begin{bmatrix} a&d&g\\b&e&h\\c&f&i \end{bmatrix}\begin{bmatrix} a&b&c\\d&e&f\\g&h&i \end{bmatrix} = -\begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} \to \begin{bmatrix} a^2+d^2+g^2 & ab+de+gh & ac+fd+gi \\ ab+de+gh & b^2+e^2+h^2 & bc+ef+hi \\ acdf+gi & bc+ef+hi & c^2+f^2+i^2 \end{bmatrix} = -\begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} = \begin{bmatrix} -1&0&0\\0&-1&0\\0&0&-1 \end{bmatrix}$

4/4

not possible b/c no solution for $a^2+d^2+g^2=-1$, $b^2+e^2+h^2=-1$, $c^2+f^2+i^2=-1$

diagonals cannot be -1

(b) [4pts.] A is a 2×2 matrix with integer entries and $\det(3A^2) = 75$.

$3^2(\det A)^2 = 75$

$\det A = \sqrt{\frac{75}{9}}$

$\begin{bmatrix} a&b\\c&d \end{bmatrix}$

$ad - bc = \sqrt{\frac{75}{9}} \to$ suppose $\boxed{b=0 \quad c=0}$

$ad = \sqrt{\frac{75}{9}} = \frac{\sqrt{75}}{3}$

$a = \frac{\sqrt{75}}{6} \quad d = 2$

$A^2 = \begin{bmatrix} \frac{\sqrt{75}}{6} & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{75}}{6} & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{75}{36} & 0 \\ 0 & 4 \end{bmatrix}$

$3A^2 = \begin{bmatrix} \frac{75}{12} & 0 \\ 0 & 12 \end{bmatrix}$ $\det 3A^2 = 75$

$A = \begin{bmatrix} \frac{\sqrt{75}}{6} & 0 \\ 0 & 2 \end{bmatrix}$

0

$\det A = ad - bc = \frac{\sqrt{75}}{3}$

$\det(3A^2) = 9(\det A)^2 = 9\left(\frac{\sqrt{75}}{3}\right)^2 = 75$

Problem 2.

(a) [5pts.] Find the least-squares solution $\vec{x}^*$ of the system

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A} \vec{x} = \underbrace{\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}}_{b}.$$

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$A^T A \vec{x}^* = A^T b$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\vec{x}^* = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 6 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{\frac{R_1}{2}} \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 0 & \frac{3}{2} & 0 \end{bmatrix} \xrightarrow{\frac{2}{3}R_2} \begin{bmatrix} 1 & \frac{1}{2} & 3 \\ 0 & 1 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

$x_1 = 3$

$x_2 = 0$ $\to \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

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(b) [5pts.] For the solution $\vec{x}^*$ you obtained in part (a), compute the error $||\vec{b} - A\vec{x}^*||$, where $\vec{b} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\|\vec{b} - A\vec{x}^*\| = \left\| \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\| \qquad \sqrt{1^2 + (-1)^2 + (-1)^2}$$

$$= \sqrt{3}$$

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$$\frac{1}{|\vec{v}_1|} = \frac{1}{\sqrt{3}}$$

Problem 3. *11pts.*

Find the QR-factorization of $M = \begin{array}{cc} \vec{v}_1 & \vec{v}_2 \\ \end{array}\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$. Make sure to justify all steps.

Using Gram Schmidt

$$\vec{u}_1 = \frac{1}{|\vec{v}_1|}\vec{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$\sqrt{\left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$\sqrt{\frac{4}{9} + \frac{1}{9} + \frac{1}{9}} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$$

$$\vec{u}_2 = \frac{1}{|\vec{w}_2|}\vec{w}_2$$

$$= \frac{3}{\sqrt{6}}\begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$\frac{2}{3} - \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{6}}{3}$

$\frac{2}{3} - \frac{4}{\sqrt{18}}$

$$Q = [\vec{u}_1 \ \vec{u}_2] = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$R = \begin{bmatrix} |\vec{u}_1| & \vec{v}_2 \cdot \vec{u}_1 \\ 0 & |\vec{u}_2| \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 2/\sqrt{3} \\ 0 & \frac{\sqrt{6}}{3} \end{bmatrix}$$

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$$M = QR$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 2/\sqrt{3} \\ 0 & \frac{\sqrt{6}}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Problem 4.

Find the determinant of the following matrices. You can use any method you want, but make sure to justify each step.

(a) [4pts.] $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. Sarrus rule $4-3+6-2-1-1$

$\begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{matrix} 1 & 2 \\ -1 & -1 \\ 2 & 1 \end{matrix}$ $\to \det A = (-1) + 4 + (-3) - (-6) - 1 - (-2)$

$\det A = -1 + 4 - 3 + 6 - 1 + 2$

$\det A = 7$

$\det \begin{bmatrix} 1 & 2 & 3 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & -3 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 7 \end{bmatrix} \to \det = 7(1)(1) = 7$

(b) [5pts.] $B = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 1 & 0 & 2 & 2 \\ 3 & 5 & 6 & 9 \\ 2 & 0 & 3 & 3 \end{bmatrix}$. Laplace $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

$18 + 36 = 54$

$\det B = -3 \det \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 9 \\ 2 & 3 & 3 \end{bmatrix} - 5 \det \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix}$

$24 + 27 = 51$

$\det B = -3(18 + 36 + \cancel{18} - 24 - 27 - \cancel{18}) - 5(12 + 4 + 0 - 0 - \cancel{12} - 3)$

$\det B = -3(3) \quad -5(1)$

$54 - 51$

$\det B = -9 - 5$

$54 - 51 = 3$

$= -14$

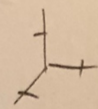
$\det B = +2 \det \begin{bmatrix} 0 & 2 & 2 \\ 5 & 6 & 9 \\ 0 & 3 & 3 \end{bmatrix} - 3 \det \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 9 \\ 2 & 3 & 3 \end{bmatrix} + 1 \begin{bmatrix} 1 & 0 & 2 \\ 3 & 5 & 9 \\ 2 & 0 & 3 \end{bmatrix} - 0$

$= 2(30 - 0 - 0 - 30) - 3(18 + 36 + \cancel{18} - 24 - 27 - \cancel{18}) + [15 + 0 + 0 - 20 - 0 - 0]$

$= 0 \quad - 3(3) \quad -5$

$= \quad -9 - 5$

$= -14$



Problem 5.

Let $\vec{v}_1 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ and $\vec{v}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$. Let P be the 4-parallelepiped defined by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$.

(a) [6pts.] Find the 4-volume of P.

$$A = \begin{bmatrix}0&0&0&1\\0&0&1&1\\0&1&1&1\\1&1&1&1\end{bmatrix} \quad A^T = A$$

$$4\text{-vol} = \sqrt{\det A^T A} \longrightarrow \sqrt{\det(A^2)} = \sqrt{(\det A)^2} = \det A$$

$$= |\det A| \quad -1$$

$$= \det \begin{bmatrix}0&0&0&1\\0&0&1&1\\0&1&1&1\\1&1&1&1\end{bmatrix}$$

~~Sarrus~~ Laplace

$$= -\det \begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix} \qquad \begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix}\begin{matrix}0&0\\0&1\\1&1\end{matrix}$$

$$= -(0+0+0-1-0-0)$$

$$= 1$$

$$= (-1)(-1)\det \begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$$

$$= 1$$

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[This problem continues from the previous page.]

Recall that $\vec{v}_1 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ and $\vec{v}_4 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$.

(b) [6pts.] Consider the linear transformation $T(\vec{x}) = \begin{bmatrix} 2 & 4 & 0 & 1 \\ 1 & -3 & -2 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 2 & -5 \end{bmatrix} \vec{x}$.

Find the 4-volume of the 4-parallelepiped defined by the vectors $T(\vec{v}_1)$, $T(\vec{v}_2)$, $T(\vec{v}_3)$ and $T(\vec{v}_4)$.

transformation matrix

from 5a

$$\text{Volume} = \text{Vol} \cdot \left|\det\left(A_{\text{transformation}}\right)\right|$$

$$\text{Vol} = 1 \cdot \left|\det\begin{bmatrix} 2&4&0&1\\1&-3&-2&0\\0&0&-1&3\\0&0&2&-5\end{bmatrix}\right|$$

$$\text{Vol} = \left|\det\begin{bmatrix} 2&4&0&1\\1&-3&-2&0\\0&0&-1&3\\0&0&2&-5\end{bmatrix}\right|$$

$$\begin{array}{rrrr} 1 & -3 & -2 & 0 \\ -1 & -2 & 0 & -\tfrac{1}{2} \\ \hline 0 & -5 & -2 & -\tfrac{1}{2} \end{array}$$

$$\begin{array}{rrrr} 0 & 0 & -2 & 6 \\ 0 & 0 & 2 & -5 \\ \hline & & 0 & 1 \end{array}$$

$$\begin{bmatrix} 2&4&0&1\\1&-3&-2&0\\0&0&-1&3\\0&0&2&-5\end{bmatrix} \xrightarrow{R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2&4&0&1\\0&-5&-2&-\frac{1}{2}\\0&0&-1&3\\0&0&2&-5\end{bmatrix} \xrightarrow{R_4 + 2R_3} \begin{bmatrix} 2&4&0&1\\0&-5&-2&-\frac{1}{2}\\0&0&-1&3\\0&0&0&1\end{bmatrix}$$

$$\det = 2(-5)(-1)(1) = 10$$

$$\text{Vol} = 10$$

6/6

Laplace

$$\det\begin{bmatrix} 2&4&0&1\\1&-3&-2&0\\0&0&-1&3\\0&0&2&-5\end{bmatrix} = 2\det\begin{bmatrix}-3&-2&0\\0&-1&3\\0&2&-5\end{bmatrix} - \det\begin{bmatrix}4&0&1\\0&-1&3\\0&2&-5\end{bmatrix}$$

$$= 2(-15-0-(-18)-0) - (+20+0+0-0-24-0)$$

$$= 2(3) - (-4)$$

$$= 10$$