Math 33A - Lectures 3 and 4
Fall 2018

Midterm 1

Instructions: You have 60 minutes to complete this exam. There are five questions, worth a total of 50 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Unless instructed otherwise, you need to justify your answers. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use the scrap pages at the end of this booklet.

Please take a moment to ensure that your booklet consists of ten pages, the last three being reserved for additional work.

Do not forget to write your full name, section and UID in the space below. For identification purposes, please sign below.

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Question	Points	Score
1	8	8
2	10	5
3	11	8
4	10	10
5	11	7
Total:	50	38

rank Null
dim

Problem 1. $\dim(\text{Im}) + \dim \ker(A) = m$

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

(a) [4pts.] A is a 3×6 matrix with rank and nullity both equal to 3.

3

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 4 & 6 \end{bmatrix}$$

4

RREF shows pivots in c_1, c_2, c_3 so rank = 3

$x_1 + x_4 + x_5 + x_6 = 0$
$x_2 + x_4 + 2x_5 + 3x_6 = 0$
$x_3 + x_4 + 4x_5 + 6x_6 = 0$

$x_1 = -x_4 - x_5 - x_6$
$x_2 = -x_4 - 2x_5 - 3x_6$
$x_3 = -x_4 - 4x_5 - 6x_6$

$$\rightarrow \begin{bmatrix} -s-t-v \\ -s-2t-3v \\ -s-4t-6v \\ s \\ t \\ v \end{bmatrix} \rightarrow s\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -2 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + v\begin{bmatrix} -1 \\ -3 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$ is for kernel, nullity

$3 + 3 = 6$

(b) [4pts.] A is a 6×3 matrix with rank and nullity both equal to 3.

4 ✓

Not possible by rank-nullity theorem. Rank = 3, & nullity = 3, so rank + nullity = 6. However there are only 3 columns & $6 \neq 3$ so it is impossible

$\frac{2x_1}{\sqrt{3}} + \frac{2x_2}{\sqrt{3}} + \frac{2x_3}{\sqrt{3}}$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $2\left(\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ $\frac{2}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$

$x - 2(x\cdot\vec{u})\vec{u}$

Problem 2.

Recall that two vectors $\vec{v}, \vec{w} \in \mathbf{R}^n$ are perpendicular if their dot product is zero: $\vec{v}\cdot\vec{w} = \vec{0}$.

(a) [5pts.] Find a nonzero matrix A such that $A\vec{x}$ is perpendicular to the vector $\vec{v} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ for every $\vec{x} \in \mathbf{R}^3$. [You do not need to justify how you found A, but you do need to show that your choice of A satisfies the prescribed condition.]

plane's equation is $x_1 + x_2 + x_3 = 0$

want: projection onto plane

$\vec{w} = [x_1\ x_2\ x_3]$ $\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\vec{w} = 0$

$w_3 = -w_1 - w_2$ $\begin{bmatrix}s\\t\\-s-t\end{bmatrix}$ → $s\begin{bmatrix}1\\0\\-1\end{bmatrix} + t\begin{bmatrix}0\\1\\-1\end{bmatrix}$

$|\vec{v}| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$\vec{u} = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$

$T(\vec{x}) = x - 2(x\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix})\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \vec{x} - \left(\frac{2x_1}{3} + \frac{2x_2}{3} + \frac{2x_3}{3}\right)$

$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} - 2\left(\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}2/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}1/3\\-2/3\\-2/3\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} - 2\left(\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}2/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}-2/3\\1/3\\-2/3\end{bmatrix}$

$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix} - 2\left(\frac{1}{\sqrt{3}}\right)\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}2/3\\2/3\\2/3\end{bmatrix} = \begin{bmatrix}-2/3\\-2/3\\1/3\end{bmatrix}$

0

$A = \begin{bmatrix}\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3}\\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}$ $\begin{bmatrix}-5/3\\-10/3\\-10/3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$ $5/3 - \frac{20}{3}$

$\begin{bmatrix}x\\y\\z\end{bmatrix} = 0$ $x+y+z=0$ $z = -x-y$ $\begin{bmatrix}s\\t\\-s-t\end{bmatrix}$ $s\begin{bmatrix}1\\0\\-1\end{bmatrix} + t\begin{bmatrix}0\\1\\-1\end{bmatrix}$

(b) [5pts.] For the matrix A you found in part (a), let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the corresponding linear transformation. Find, with justification, a basis of the image of T.

$-2\left(\frac{2}{3}\right) = \frac{-4}{3}$

Basis of im of T

$\begin{bmatrix}\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3}\\ -2/3 & \frac{1}{3} & -\frac{2}{3}\\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix} \xrightarrow{3R_1} \begin{bmatrix}1 & -2 & -2\\ -2/3 & 1/3 & -2/3\\ -2/3 & -2/3 & 1/3\end{bmatrix} \begin{matrix}R_2 + \frac{2}{3}R_1\\ R_3 + \frac{2}{3}R_1\end{matrix} \to \begin{bmatrix}1 & -2 & -2\\ 0 & -1 & -2\\ 0 & -2 & -1\end{bmatrix} \frac{R_2}{-1} =$

$\begin{bmatrix}1 & -2 & -2\\ 0 & 1 & 2\\ 0 & -2 & -1\end{bmatrix} \begin{matrix}R_1 + 2R_2\\ R_3 + 2R_2\end{matrix} \to \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 0 & 3\end{bmatrix} \frac{R_3}{3} \Rightarrow \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix} \begin{matrix}R_1 - 2R_3\\ R_2 - 2R_3\end{matrix} \to$

$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ Pivot in all col so basis of im of T is $\left\{\begin{bmatrix}\frac{1}{3}\\-2/3\\-2/3\end{bmatrix}, \begin{bmatrix}-2/3\\1/3\\-2/3\end{bmatrix}, \begin{bmatrix}2/3\\-2/3\\1/3\end{bmatrix}\right\}$

5

$\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\end{bmatrix}$

Problem 3.

Let $\vec{x}, \vec{y}$ be two nonzero vectors in $\mathbf{R}^n$. Consider the set

$$V = \{\vec{v} \in \mathbf{R}^n \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}.$$

$\vec{v}_1 \vec{x} = \vec{v}_1 \vec{y}$ & $\vec{v}_2 \vec{x} =$

(a) [3pts.] Prove that V is a subspace of $\mathbf{R}^n$.

$0 \in V$ is true why?

have $\vec{v}_1 \in \mathbb{R}^n$ & $\vec{v}_2 \in \mathbb{R}^n \to (\vec{v}_1 + \vec{v}_2) \cdot \vec{x} = (\vec{v}_1 + v_2)\vec{y} \to \vec{v}_1\vec{x} + \vec{v}_2\vec{x} = \vec{v}_1\vec{y} + \vec{v}_2$ ✓

$1 = 1$ true

$(k\vec{v}) \to k \in \mathbb{R}^n \to k\vec{v} \cdot x = k\vec{v} \cdot \vec{y}$ is true for all k

$k(\vec{v} \cdot y) = k(\vec{v} \cdot \vec{y})$ ✓

$1 = 1$

3/3

(b) [3pts.] Let now $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ be vectors in $\mathbf{R}^4$ and let V be defined as above.

Show that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ belong to V.

3/3

$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

$2 + 1 + 1 = 1 + 2 + 1$

$4 = 4$

so $\vec{v}_1$ belongs to V

$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

$1 = 2 - 1$

$1 = 1$

$\vec{v}_2$ belongs to V

$\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$

$2 - 1 - 1 = 1 - 2 + 1$

$0 = 0$

so $\vec{v}_3$ belongs to V

This problem continues from the previous page. Recall that $\vec{x}, \vec{y}, \vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are defined in question (b), and $V = \{\vec{v} \in \mathbf{R}^4 \text{ such that } \vec{v}\cdot\vec{x} = \vec{v}\cdot\vec{y}\}$.

(c) [5pts.] Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V.

$v_1, v_2, v_3 \to \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}$ are all in V by part b

so v_1, v_2, v_3 ~~span~~ V

No, you need to show that

$x\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix} + y\begin{bmatrix}0\\1\\1\\0\end{bmatrix} + z\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix} = 0$ only when $x = y = z = 0$

so linearly independent

$\begin{bmatrix}1 & 0 & 1 & | & 0\\ -1 & 1 & -1 & | & 0\\ -1 & 1 & -1 & | & 0\\ 1 & 0 & -1 & | & 0\end{bmatrix}$ $R_2 - R_1$, $R_3 + R_1$, $R_4 - R_1$

each are in $\mathbb{R}_4$

$\begin{bmatrix}1 & 0 & 1 & | & 0\\ 0 & 0 & -2 & | & 0\\ 0 & 1 & 0 & | & 0\\ 0 & 0 & -2 & | & 0\end{bmatrix}$ swap R_2 & R_3

2/5

$\begin{bmatrix}1 & 0 & 1 & | & 0\\ 0 & 1 & 0 & | & 0\\ 0 & 0 & -2 & | & 0\\ 0 & 0 & -2 & | & 0\end{bmatrix}$ $\frac{R_3}{-2}$

$\therefore \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ ~~basis of V~~

$\begin{bmatrix}1 & 0 & 1 & | & 0\\ 0 & 1 & 0 & | & 0\\ 0 & 0 & 1 & | & 0\\ 0 & 0 & -2 & | & 0\end{bmatrix}$ $R_1 - R_3$, $R_4 + 2R_3$

$\begin{bmatrix}1 & 0 & 0 & | & 0\\ 0 & 1 & 0 & | & 0\\ 0 & 0 & 1 & | & 0\\ 0 & 0 & 0 & | & 0\end{bmatrix}$ $x = 0$, $y = 0$, $z = 0$

Problem 4.

(a) [5pts.] Using row-reduction find, if it exists, the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 5 & 8 & 0 & 1 & 0 \\ 3 & 6 & 9 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array} \to \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right] \frac{R_2}{-3} \to \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 2 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1-4R_2 \\ R_3+6R_2 \end{array} \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & (1+\frac{8}{3}) & -\frac{4}{3} & 0 \\ 0 & 1 & 2 & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & -7 & 2 & 1 \end{array}\right]$$

Inverse does not exist b/c RREF is not I_3 ✓

(b) [5pts.] Let A be the matrix defined in (a). Find all solutions $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of the system $A\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. [There could be none.]

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 1 \end{array}\right] \begin{array}{l} R_2-2R_2 \\ R_3-3R_3 \end{array} \to \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & -6 & -12 & 4 \end{array}\right] \frac{R_2}{-3} \to \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & 1 & 2 & -\frac{2}{3} \\ 0 & -6 & -12 & 4 \end{array}\right] \begin{array}{l} R_1-4R_2 \\ R_3+6R_2 \end{array} \to$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & (-1+\frac{8}{3}) \\ 0 & 1 & 2 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x - z = \frac{5}{3} \to x = \frac{5}{3} + z$

$y + 2z = -\frac{2}{3} \qquad y = -\frac{2}{3} - 2z$

$z = t$

$$\begin{bmatrix} \frac{5}{3} + t \\ -\frac{2}{3} - 2t \\ t \end{bmatrix} \quad \forall t \in \mathbb{R}$$

10

unit normal $= \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}$ $\langle 1,-1,1\rangle$ is orthogonal

$\vec{x} - 2(\vec{x}\cdot\vec{u})\vec{u}$

Problem 5.

Let P be the plane in $\mathbf{R}^3$ given by the equation $x - y + z = 0$.

(a) [5pts.] Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the reflection across the plane P. Find the matrix of T with respect to the standard basis of $\mathbf{R}^3$.

$T = x - 2\left(\vec{x}\cdot\begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}\right)\begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix}$ $2\left(\frac{1}{\sqrt{3}}\right)\begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{2}{3}\end{bmatrix}$

$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{2}{3}\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}$ $2\left(-\frac{1}{\sqrt{3}}\right)\begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix} = \begin{bmatrix}-2/3\\2/3\\-2/3\end{bmatrix}$

$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}-2/3\\2/3\\-2/3\end{bmatrix} = \begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix}$ $2\left(\frac{1}{\sqrt{3}}\right)\begin{bmatrix}\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{bmatrix} = \begin{bmatrix}2/3\\-2/3\\2/3\end{bmatrix}$

$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}2/3\\-2/3\\2/3\end{bmatrix} = \begin{bmatrix}-2/3\\2/3\\1/3\end{bmatrix}$

Matrix of $T = \begin{bmatrix}\frac{1}{3} & \frac{2}{3} & -\frac{2}{3}\\\frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\-\frac{2}{3} & \frac{2}{3} & \frac{1}{3}\end{bmatrix}$

(b) [6pts.] Find a basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbf{R}^3$ such that the $\mathfrak{B}$-matrix of T is diagonal. Write down the $\mathfrak{B}$-matrix of T.

$\vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

not ⊥

$T(\vec{v}_1) = A\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} = \vec{v}_1 \to \begin{bmatrix}1\\0\\0\end{bmatrix}_{\mathfrak{B}}$ $\mathfrak{B}$ matrix $= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$T(\vec{v}_2) = A\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix} = \vec{v}_2 \to \begin{bmatrix}0\\1\\0\end{bmatrix}_{\mathfrak{B}}$

2/6

$T(\vec{v}_3) = A\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} = \vec{v}_3 \to \begin{bmatrix}0\\0\\1\end{bmatrix}_{\mathfrak{B}}$