Math 33A - Lectures 3 and 4
Fall 2018

Midterm 1

Instructions: You have 60 minutes to complete this exam. There are five questions, worth a total of 50 points. This test is closed book and closed notes. No calculator is allowed.

For full credit show all of your work legibly. Unless instructed otherwise, you need to justify your answers. Please write your solutions in the space below the questions; INDICATE if you go over the page and/or use the scrap pages at the end of this booklet.

Please take a moment to ensure that your booklet consists of ten pages, the last three being reserved for additional work.

Do not forget to write your full name, section and UID in the space below. For identification purposes, please sign below.

Full Name: [redacted]
Student ID [redacted]
Lectur[redacted]
Sectio[redacted]

Signature: [redacted]

Question	Points	Score
1	8	8
2	10	7
3	11	5
4	10	9
5	11	6
Total:	50	35

Problem 1.

For each of the following sentences, give an example of a matrix A with the following properties, or explain why it is impossible.

(a) [4pts.] A is a 3×6 matrix with rank and nullity both equal to 3.

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}$$

4

(b) [4pts.] A is a 6×3 matrix with rank and nullity both equal to 3.

4

Not possible, rank nullity theorem

$\dim(\ker) + \dim(\text{rank}) = n$

$3 + 3 \neq 3$

Problem 2.

Recall that two vectors $\vec{v}, \vec{w} \in \mathbf{R}^n$ are perpendicular if their dot product is zero: $\vec{v} \cdot \vec{w} = \vec{0}$.

(a) [5pts.] Find a nonzero matrix A such that $A\vec{x}$ is perpendicular to the vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for every $\vec{x} \in \mathbf{R}^3$. [You do not need to justify how you found A, but you do need to show that your choice of A satisfies the prescribed condition.]

$\vec{u} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$T(\vec{x}) = \vec{x} - (\vec{x} \cdot \vec{u})\vec{u}$

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \left(\frac{1}{\sqrt{3}}\right)\left[\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]$

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

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$$A = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \to A\vec{x} \cdot \vec{v} = \left(x_1 \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} + x_2 \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} + x_3 \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \right) \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Proof: $\frac{2}{3}x_1 - \frac{1}{3}x_1 - \frac{1}{3}x_1 + -\frac{1}{3}x_2 + \frac{2}{3}x_2 - \frac{1}{3}x_2 - \frac{1}{3}x_3 - \frac{1}{3}x_3 + \frac{2}{3}x_3$

$0 + 0 + 0 = 0$ ✓

(b) [5pts.] For the matrix A you found in part (a), let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the corresponding linear transformation. Find, with justification, a basis of the image of T.

A was created with the ⊥ component of the vector $\vec{x}$ to the line spanned by $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Geometrically, this means the image must be the plane ⊥ to this line, intersecting at the origin.

2

Problem 3.

Let $\vec{x}, \vec{y}$ be two nonzero vectors in $\mathbf{R}^n$. Consider the set

$$V = \{\vec{v} \in \mathbf{R}^n \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}.$$

(a) [3pts.] Prove that V is a subspace of $\mathbf{R}^n$.

let $\vec{v}_1, \vec{v}_2 \in V$

check, $\vec{v}_1 + \vec{v}_2$ still in V:

$(\vec{v}_1 + \vec{v}_2) \cdot \vec{x} \stackrel{?}{=} (\vec{v}_1 + \vec{v}_2) \cdot \vec{y}$ 1/3

dot product is distributive this works

Prove they are unique: ?

$a_1 V_1 + \dots a_n V_n = 0$, only when $a_1 \dots a_n = 0$

(b) [3pts.] Let now $\vec{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ be vectors in $\mathbf{R}^4$ and let V be defined as above.

Show that $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$ belong to V.

linear combinators

1) $V_1 \cdot x = V_1 \cdot Y$

$$\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

$2+1+1 \qquad 1+2+1$

$4 = 4$ ✓

$V_2 \cdot X$, $\cancel{V_2} \cdot Y$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

$1 = 2-1$

$1 = 1$ ✓

$V_3 \cdot X$, $V_3 \cdot Y$

$$\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

$2-1-1 = 1-2+1$

$0 = 0$

✓

3/3

This problem continues from the previous page. Recall that $\vec{x}, \vec{y}, \vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are defined in question (b), and $V = \{\vec{v} \in \mathbf{R}^4 \text{ such that } \vec{v} \cdot \vec{x} = \vec{v} \cdot \vec{y}\}$.

(c) [5pts.] Prove that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis of V.

$V_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad V_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$

$a_1 V_1 + a_2 V_2 + a_3 V_3 = b_1 V_1 + b_2 V_2 + b_3 V_3$ (2 different sets of constants representing same vector)

$V_1(a_1 - b_1) + V_2(a_2 - b_2) + V_3(a_3 - b_3) = 0$

The only way ~~to = 0~~ is if $a_1 = b_1, a_2 = b_2, a_3 = b_3$, they are unique and linearly independent.

Needs proof

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

Because they are linearly independent they are all pivot columns formed by vectors in V and it must be a ~~basis of it~~.

Check they span V

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Problem 4.

(a) [5pts.] Using row-reduction find, if it exists, the inverse of the matrix

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 2 & 5 & 8 & 0 & 1 & 0 \\ 3 & 6 & 9 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ -2R_1 \\ -3R_1 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & -3 & -6 & -2 & 1 & 0 \\ 0 & -6 & -12 & -3 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \div -3 \\ \div -6 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 7 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 2 & \frac{1}{2} & 0 & -\frac{1}{6} \end{array}\right]$$

these rows are inconsistent, inverse does not exist.

this does not became I_3

(b) [5pts.] Let A be the matrix defined in (a). Find all solutions $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of the system

$A\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. [There could be none.]

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 1 \end{array}\right] \begin{array}{c} \\ -2R_1 \\ -3R_1 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & -6 & -12 & 4 \end{array}\right] \begin{array}{c} \\ \\ \div 2 \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & -3 & -6 & 2 \end{array}\right] \begin{array}{c} \\ \\ -R2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & -3 & -6 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \div -3 \rightarrow \left[\begin{array}{ccc|c} 1 & 4 & 7 & -1 \\ 0 & 1 & 2 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{-4R2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & \frac{5}{3} \\ 0 & 1 & 2 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x - z = \frac{5}{3}$, $y + 2z = -\frac{2}{3}$

$z = x - \frac{5}{3}$ $\quad z = -\frac{2}{3} - y$

$$\begin{bmatrix} q + \frac{5}{3} \\ 2q - \frac{2}{3} \\ q \end{bmatrix}$$

9

component // to normal

Problem 5.

Let P be the plane in $\mathbf{R}^3$ given by the equation $x - y + z = 0$.

(a) [5pts.] Let $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^3$ be the reflection across the plane P. Find the matrix of T with respect to the standard basis of $\mathbf{R}^3$.

where u from normal $= \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{3}}$

$T(\vec{x}) = (\vec{u} \cdot \vec{x}) \vec{u}$

$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$

$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$

$\therefore T_{matrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$

3/5

(b) [6pts.] Find a basis $\mathfrak{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbf{R}^3$ such that the $\mathfrak{B}$-matrix of T is diagonal. Write down the $\mathfrak{B}$-matrix of T.

needs to be scaling itself by a constant,

$k(T(\vec{x})) = T(k\vec{x})$

3/6

$(u \cdot k\vec{x})\vec{u}$

$\left(\frac{1}{\sqrt{3}} kx_1 - \frac{1}{\sqrt{3}} kx_2 + \frac{1}{\sqrt{3}} kx_3\right) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

$\frac{k}{\sqrt{3}} (x_1 - x_2 + x_3) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$k(x_1 - x_2 + x_3) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ pick $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \to \begin{bmatrix} x_1 - x_2 + x_3 \\ -x_1 + x_2 - x_3 \\ x_1 - x_2 + x_3 \end{bmatrix}$ 3 0 -3 0 3 0