NAME [redacted]

Hour Exam I – Mathematics 33A

October 15, 2010

1	10
2	3
3	10
4	10
5	7
	40

10

1. Find the inverse for the matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 2 \\ 0 & -1 & 2 \end{pmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \Rightarrow 2 \\ \Rightarrow 1 \\ \Rightarrow 1+3 \end{array} \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \Rightarrow 1-2(3) \\ \Rightarrow 2+3 \\ \Rightarrow 3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & -2 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -2 & 1 & -2 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

3. Suppose that A is an invertible $n \times n$ matrix, B is an $n \times r$ matrix, $\vec{x}$ is an n-component vector, and $\vec{y}$ is an r-component vector. If

$$(A\ B)\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix} = \vec{b},$$

find a formula for $\vec{x}$ in terms of A^{-1}, B, $\vec{b}$ and $\vec{y}$.

$$A\vec{x} + B\vec{y} = \vec{b}$$

$$A\vec{x} = \vec{b} - B\vec{y}$$

$$A^{-1}A\vec{x} = A^{-1}[\vec{b} - B\vec{y}]$$

$$I\vec{x} = A^{-1}\vec{b} - A^{-1}B\vec{y}$$

$$\vec{x} = A^{-1}\vec{b} - A^{-1}B\vec{y}$$

10

4. (a) (5 pts.) Let $P(\vec{x})$ be the orthogonal projection of $\vec{x} \in \mathbb{R}^2$ onto the line $x_2 = 2x_1$. Find the 2×2 matrix A such that $P(\vec{x}) = A\vec{x}$.

$y = 2x \quad A = \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$

$\text{proj}_{\vec{u}_L} \vec{x} = (\vec{x} \cdot \vec{u})\vec{u} \qquad V: \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \vec{u}_L : \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

$X = \underset{\vec{i}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \ \& \ \underset{\vec{j}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

1) $(\vec{i} \cdot \vec{u})\vec{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ 2/5 \end{bmatrix}$

2) $(\vec{j} \cdot \vec{u})\vec{u} = \frac{2}{\sqrt{5}} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/5 \\ 4/5 \end{bmatrix}$

$A = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$

$a = \frac{1}{\sqrt{5}}$

$b = \frac{2}{\sqrt{5}}$

(b) (5 pts.) Let $R(\vec{x})$ be the reflection $\vec{x} \in \mathbb{R}^2$ about the line in part (a). Find the 2×2 matrix B such that $R(\vec{x}) = B\vec{x}$.

$B = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$

$\text{ref}_L \vec{X} = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{X} \qquad u = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$

$\vec{X}_{\vec{i}} : \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow (\vec{x} \cdot \vec{u})\vec{u} = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$

$\vec{X}_{\vec{j}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow (\vec{x} \cdot \vec{u})\vec{u} = \begin{bmatrix} 2/5 \\ 4/5 \end{bmatrix}$

$\text{ref}_L \vec{x}_{\vec{i}} = 2\begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/5 - 5/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$

$\text{ref}_L \vec{x}_{\vec{j}} = 2\begin{bmatrix} 2/5 \\ 4/5 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 8/5 - 5/5 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$

$T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$

$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3/5 \\ 4/5 \end{bmatrix}$

$\longrightarrow B = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix} \qquad a = -3/5$

$b = 4/5$

5. Let $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{pmatrix} 1 & 3 & -2 & 2 \\ 3 & 9 & -6 & 6 \end{pmatrix}$$

(a) (2 pts.) What is the reduced row echelon form of A?

$$\text{rref}(A) = \begin{pmatrix} 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

✓

(b) (2 pts.) What is the rank of A?

1

✓

(c) (3 pts.) Find a set of vectors (as small a set as is possible) whose linear combinations are the image of T.

$$\rightarrow \quad x_1\begin{bmatrix}1\\3\end{bmatrix} + \underbrace{x_2\begin{bmatrix}3\\9\end{bmatrix} + x_3\begin{bmatrix}-2\\-6\end{bmatrix} + x_4\begin{bmatrix}2\\6\end{bmatrix}}_{\text{redundant}}$$

$$\boxed{\vec{V}_I = \begin{bmatrix}1\\3\end{bmatrix}}$$

✓

(d) (3 pts.) Find a set of vectors $\underbrace{\text{(as small a set as is possible)}}_{\text{no redundancy}}$ whose linear combinations are the kernel of T.

makes line $\rightarrow$ $\vec{0}$ vector

$$\vec{V}_K = \vec{0}$$

✗