Problem 1. Below is a list of statements. Decide which are true and which are false. On the left of each, write "TRUE" or "FALSE" in capital letters. You must also write your answer ("TRUE" or "FALSE" in capital letters) on the front page of the exam.

There is no partial credit on this problem.

(A) (2 points) Suppose A is $m \times n$. The vector $A\vec{x}$ for $\vec{x} \in \mathbb{R}^n$ is a linear combination of the rows of A.
Solution: False. Replace rows with columns to get a true statement.

(B) (2 points) Every square matrix is invertible.
Solution: False. The zero matrix is a counter example.

(C) (2 points) Suppose A is $m \times n$, and $A\vec{x} = \vec{b}$ has a unique solution for some vector $\vec{b} \in \mathbb{R}^m$. Then $n > m$.
Solution: False. If there's a unique solution for some $\vec{b}$, then $\text{rref}(A)$ must have a pivot in every column. Hence there must be at least as many rows as columns, so $m \geq n$.

(D) (2 points) Suppose A is $m \times n$, with $n > m$. Then $\ker(A) = \{0\}$.
Solution: False. If $n > m$, then there will be columns without pivots in $\text{rref}(A)$, which means there are free variables.

(E) (2 points) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one, then $\ker(T) = \{\vec{0}\}$.
Solution: True. Suppose $\vec{x} \in \ker(T)$, so $T\vec{x} = 0$. Then $T\vec{x} = T\vec{0}$, so since T is one-to-one, $\vec{x} = \vec{0}$. Hence $\ker(T) \subset \{\vec{0}\}$. The other containment is trivial.
Remark: There was a second version of the test with 2 similar questions, and 3 of the same in different order. We give those solutions below.

(F) (2 points) Suppose A is $m \times n$, with $\ker(A) = \{0\}$. Then $m = n$.
Solution: False. All we know is that there are at least as many rows as columns, but there could be more! For instance,

$$A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

satisfies $\ker(A) = \vec{0}$.

(G) (2 points) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto, then $\ker(T) = \{\vec{0}\}$. **Solution:** False. To get a true statement, replace onto with one-to-one. There are plenty of counter examples. For instance, $T = L_A : \mathbb{R}^2 \to \mathbb{R}$ for

$$A = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

satisfies T is onto, but $\ker(T)$ is spanned by

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Problem 2. (*10 points*) You must show all work to get partial credit.

(a) (5 points) Use Gaussian elimination to compute the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

Solution: We perform Gaussian elimination on the following matrix:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right)$$
$$\longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right).$$

One now verifies:

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = I_2.$$

(b) (5 points) Find a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$, but $BA \neq I_3$. *Hint: You can do this only using* 0*'s and* 1*'s for the entries of* A *and* B.

Solution: One example that works is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Problem 3. (*10 points*) You must show all work to get partial credit.

Consider the 4×5 matrix $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

(a) (5 points) Find a set of vectors in $\mathbb{R}^5$ which spans $\ker(A)$.

Solution: The free variables correspond to the columns without pivots. We set $x_2 = s$ and $x_5 = t$. We then have the following equations:

$$x_1 + 2s + 3t = 0$$
$$x_3 + 2t = 0$$
$$x_4 + t = 0.$$

This means that any vector in the kernel is of the form

$$\vec{x} = \begin{pmatrix} -2s-3t \\ s \\ -2t \\ -t \\ t \end{pmatrix} = s\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 0 \\ -21 \\ -1 \\ 1 \end{pmatrix}.$$

This means that a spanning set for $\ker(A)$ is given by the set

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -21 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

(b) (3 points) Find a set of vectors in $\mathbb{R}^4$ which spans $\text{im}(A)$.
Solution: The cheap answer is to take all the columns of A. But this won't get you the remaining points in part (c). We see A is already in reduced row echelon form. By inspection, we can see that the second and fifth columns are redundant, and can be ignored without affecting the span of the columns. Hence a spanning set is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(c) (2 points) If you answered both part (a) and part (b) correctly, you get the final 2 points for this question if the number of vectors you found in part (a) plus the number of vectors you found in part (b) equals 5.
Solution: We verify that $2+3=5$. Hooray! More importantly, this means that we have actually found bases for $\ker(A)$ and $\text{im}(A) = \text{CS}(A)$, not just spanning sets. The Rank Nullity Theorem tells us that the rank (the dimension of $\text{im}(A)$) plus the nullity (the dimension of $\ker(A)$ equals the number of columns. Since A is already in reduced row echelon form, we can see that the rank of A is 3, so a basis of $\text{im}(A)$ has 3 elements, and a basis of $\ker(A)$ has 2 elements.

Problem 4. (*10 points*) Suppose we have j vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j \in \mathbb{R}^n$ and k vectors $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k \in \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j\}$. Prove that any linear combination of $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k$ is also a linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j$.

Note: You are being asked to prove $V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_j\}$ *is closed under taking linear combinations. You may not assume* V *is a subspace in this question, since that would*

beg the question!
Solution: Suppose that we have a linear combination

$$\vec{w} = \sum_{i=1}^{k} c_i \vec{w}_i \qquad c_i \in \mathbb{R}.$$

For each $i = 1, \ldots, n$, there are $a_{i,\ell} \in \mathbb{R}$ for $\ell = 1, \ldots, j$ such that

$$\vec{w}_i = \sum_{\ell=1}^{j} a_{i,\ell} \vec{v}_\ell \qquad \text{for all } \ell = 1, \ldots, j.$$

We now see that

$$\begin{aligned}
\vec{w} &= \sum_{i=1}^{k} c_i \vec{w}_i \\
&= \sum_{i=1}^{k} c_i \left(\sum_{\ell=1}^{j} a_{i,\ell} \vec{v}_\ell \right) \\
&= \sum_{i=1}^{k} \sum_{\ell=1}^{j} c_i a_{i,\ell} \vec{v}_\ell \\
&= \sum_{\ell=1}^{j} \sum_{i=1}^{k} c_i a_{i,\ell} \vec{v}_\ell \\
&= \sum_{\ell=1}^{j} \underbrace{\left(\sum_{i=1}^{k} c_i a_{i,\ell} \right)}_{b_\ell} \vec{v}_\ell \\
&= \sum_{\ell=1}^{j} b_\ell \vec{v}_\ell .
\end{aligned}$$

Thus $\vec{w}$ is a linear combination of $\vec{v}_1, \ldots, \vec{v}_j$.