Math 33A/2 Linear Algebra with Applications
Exam II Solution
Friday, November 16, 2012
Prof. Ouellette

Name: ____ **ID#**: ________________

Instructions: You have 50 minutes to complete the exam. Do not open this exam until the professor instructs you to do so. When instructed to begin, solve the following problems in the spaces provided. *To receive full credit, you must show all of your work and explain your reasoning.* Partial credit will be awarded. You may cite without proof any facts that were either assigned as homework problems or covered in lecture unless the facts are explicitly stated as exam problems. You may not consult any people, books, notes, exams, quizzes, calculators, electronic devices, or any other aids during the exam.

Check the box to the left of the discussion section that you are enrolled in:

	2A T Bunche 3211		2B R Kaufman 101
	2C T Dodd 78		2D R Fowler A103B
	2E T Geology 4660		2F R Kinsey Pavillion 1220B

Please do not write in the space below.

	Possible Points	Your Score
Problem 1	15	15
Problem 2	15	15
Problem 3	15	15
Problem 4	15	15
Problem 5	20	20
Total Score:	80	80

GOOD LUCK!

1. **Matrix Invertibility.** [15pts.] Determine whether or not the matrix is invertible. If it is, compute its inverse. If not, explain why.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 0 \\ 4 & 0 & 3 \end{bmatrix}$$

Solution. We consider the augmented matrix $[A|I_3]$ and compute $RREF[A|I_3]$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 0 & 3 & 0 & 0 & 1 \end{array}\right]$$

Add -3(Row 1) to Row 2 and -4(Row 1) to (Row 3) to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 & 1 & 0 \\ 0 & 0 & -1 & -4 & 0 & 1 \end{array}\right]$$

Scale Row 3 by -1 to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & -3 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 & -1 \end{array}\right]$$

Add 3(Row 3) to Row 2 and -1(Row 3) to (Row 1) to get

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 0 & 1 \\ 0 & 1 & 0 & 9 & 1 & -3 \\ 0 & 0 & 1 & 4 & 0 & -1 \end{array}\right]$$

Thus A is invertible with inverse

$$A^{-1} = \begin{bmatrix} -3 & 0 & 1 \\ 9 & 1 & -3 \\ 4 & 0 & -1 \end{bmatrix}$$

2. **Kernel and Image.** Consider the matrices

$$A = \begin{bmatrix} -1 & 0 & 2 & 9 \\ 1 & 0 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \text{ and } RREF(A) = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) [8pts.] Compute a basis for Ker A.

Solution.

$$\text{Ker } A = \text{Ker } RREF(A)$$

Let $x_2 = r$ and $x_4 = s$ be real parameters. Then Ker A is the set of all vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Hence, a basis for Ker (A) is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(b) [7pts.] Compute a basis for Im A.

Solution. Note that the columns of A span Im A. To find a basis for Im A, we need only remove the redundant column vectors. The redundant columns of $RREF(A)$ (and of A) are the second and fourth columns. Thus a basis for Im A is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$$

3. **Subspaces and Dimension.**

(a) [7pts.] Consider the the subspace W of $\mathbb{R}^3$ having the following basis

$$\beta = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \right\}$$

Is the following vector in W? Explain why or why not.

$$\begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}$$

Solution. The vector is in W if and only if the following linear system is consistent

$$\left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 4 & 8 & 9 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

The system is inconsistent due to the last row, so the answer is "No".

(b) [8pts.] Suppose V and W are subspaces of $\mathbb{R}^n$ such that V is contained in W. Prove that if $\dim(V) = \dim(W)$, then $V = W$.
Hint: What has to be true about any basis for V?

Solution. Any linearly independent collection of vectors in W of size $\dim(W)$ must be basis for W. A basis for V is such a collection since V is contained in W and $\dim(V) = \dim(W)$. So a basis for V must also be a basis for W, forcing W to be contained in V.

4. **Matrices of a Linear Transformation and Coordinates.**

Consider the linear transformation $T(\overrightarrow{\mathbf{x}}) = A\overrightarrow{\mathbf{x}}$ where $A = \begin{bmatrix} 7 & 9 \\ -4 & -5 \end{bmatrix}$

Let $\beta = \left\{ \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, which is a basis for $\mathbb{R}^2$.

(a) [10pts.] Compute the β-matrix $B = [T]_\beta^\beta$.

Solution.

$$\begin{aligned} S = [I]_\beta^{\mathcal{E}} &= \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \\ S^{-1} = [I]_{\mathcal{E}}^\beta &= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \\ B = S^{-1}AS &= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 7 & 9 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Alternatively, we construct B column-by-column.

$$T\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} = 1\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$T\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 1\begin{bmatrix} 3 \\ -2 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

so

$$B = \left[\left[T\begin{bmatrix} 3 \\ -2 \end{bmatrix} \right]_\beta \left[T\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right]_\beta \right] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(b) [5pts.] Compute $\overrightarrow{\mathbf{x}}$ when its coordinate vector with respect to β above is

$$[\overrightarrow{\mathbf{x}}]_\beta = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Solution.

$$\overrightarrow{\mathbf{x}} = S[\overrightarrow{\mathbf{x}}]_\beta = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$

5. **Gram-Schmidt Orthonormalization and QR Factorization.**

(a) [12pts.] Perform the Gram-Schmidt Process on the following basis $\mathcal{B} = \{\overrightarrow{\mathbf{w_1}}, \overrightarrow{\mathbf{w_2}}\}$ to construct an orthornomal basis $\mathcal{U} = \{\overrightarrow{\mathbf{u_1}}, \overrightarrow{\mathbf{u_2}}\}$.

$$\overrightarrow{\mathbf{w_1}} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \overrightarrow{\mathbf{w_2}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$

Solution. Performing Gram-Schmidt:

$$\begin{aligned}
\left\|\overrightarrow{\mathbf{w_1}}\right\| &= \sqrt{2^2 + 1^2 + 2^2} = 3 \\
\overrightarrow{\mathbf{u_1}} &= \frac{\overrightarrow{\mathbf{w_1}}}{\left\|\overrightarrow{\mathbf{w_1}}\right\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \\
(\overrightarrow{\mathbf{u_1}} \cdot \overrightarrow{\mathbf{w_2}}) &= \frac{2}{3}(1) + \frac{1}{3}(1) + \frac{2}{3}(0) = 1 \\
\overrightarrow{\mathbf{w_2^{\perp}}} &= \overrightarrow{\mathbf{w_2}} - (\overrightarrow{\mathbf{u_1}} \cdot \overrightarrow{\mathbf{w_2}})\overrightarrow{\mathbf{u_1}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 1\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \\
\left\|\overrightarrow{\mathbf{w_2^{\perp}}}\right\| &= 1 \\
\overrightarrow{\mathbf{u_2}} &= \frac{\overrightarrow{\mathbf{w_2^{\perp}}}}{\left\|\overrightarrow{\mathbf{w_2^{\perp}}}\right\|} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}
\end{aligned}$$

(b) [8pts.] Find the QR factorization of M where

$$M = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

Solution. From (a), $M = QR$ is

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \\ 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$