Math 33A/2 Linear Algebra with Applications
Exam I Solution
Monday, October 22, 2012
Prof. Ouellette

Name: ____ **ID#**: ________________

Instructions: You have 50 minutes to complete the exam. Do not open this exam until the professor instructs you to do so. When instructed to begin, solve the following problems in the spaces provided. *To receive full credit, you must show all of your work and explain your reasoning.* Partial credit will be awarded. You may cite without proof any facts that were either assigned as homework problems or covered in lecture unless the facts are explicitly stated as exam problems. You may not consult any people, books, notes, exams, quizzes, calculators, electronic devices, or any other aids during the exam.

Check the box to the left of the discussion section that you are enrolled in:

	2A T Bunche 3211		2B R Kaufman 101
	2C T Dodd 78		2D R Fowler A103B
	2E T Geology 4660		2F R Kinsey Pavillion 1220B

Please do not write in the space below.

	Possible Points	Your Score
Problem 1	15	15
Problem 2	10	10
Problem 3	10	10
Problem 4	20	20
Problem 5	15	15
Total Score:	70	70

GOOD LUCK!

1. **Solving Linear Systems.** [15pts.] Solve the following linear system using Gauss-Jordan elimination on the augmented matrix. To receive full credit, you must do the correct steps in order.

$$\begin{aligned} y + 3z &= 1 \\ 2x \qquad + 4z &= 2 \\ 5x \qquad + 10z &= 5 \end{aligned}$$

Solution. We form the augmented matrix $[A|\overrightarrow{\mathbf{b}}]$ of the linear system:

$$\left[\begin{array}{ccr|c} 0 & 1 & 3 & 1 \\ 2 & 0 & 4 & 2 \\ 5 & 0 & 10 & 5 \end{array}\right]$$

Row 1 is the current row.

- Step 1: Swap current row with the first row having the leftmost nonzero entry.

$$\left[\begin{array}{ccr|c} 2 & 0 & 4 & 2 \\ 0 & 1 & 3 & 1 \\ 5 & 0 & 10 & 5 \end{array}\right]$$

- Step 2: Scale to get a leading 1 in current row.

$$\left[\begin{array}{ccr|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 5 & 0 & 10 & 5 \end{array}\right]$$

- Step 3: Add multiples of current row to other rows to eliminate entries above and below the pivot.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = t$ where t is a real parameter. The general parametrized solution vector is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t+1 \\ -3t+1 \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2. **Number of Solutions of a Linear System.** [10pts.]

Consider the linear system whose augmented matrix $[A|\overrightarrow{\mathbf{b}}]$ is given by:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(a) [2pts.] Is this matrix in reduced row echelon form? If not, compute $RREF[A|\overrightarrow{\mathbf{b}}]$.

Solution. It is not in RREF.

$$RREF[A|\overrightarrow{\mathbf{b}}] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(b) [3pts.] Compute rank (A), the rank of the coefficient matrix $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution.

$$\text{rank } (A) = \text{ number of leading ones in } RREF(A) = 2$$

(c) [3pts.] State and use a formula to compute the number of free variables of the system.

Solution. If m is the total number of variables both free and fixed, then

$$\text{number of free variables } = m - \text{rank } (A) = 3 - 2 = 1$$

(d) [2pts.] Does the system have any solutions? Explain why or why not.

Solution. This system has no solution since it is inconsistent. It contains an inconsistent row $[0...0|1]$.

3. **Linear Transformations.** [10pts.]

(a) [5pts.] Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 - x_2 \\ \quad x_1 + x_2 + 1 \end{bmatrix}$$

Is T linear? If so, find the matrix of T to prove it. If not, explain why not.

Solution. T is not linear since linear transformations must map $\overrightarrow{\mathbf{0}}$ to $\overrightarrow{\mathbf{0}}$:

$$T\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) [5pts.] Find a 2×2 matrix A such that

$$A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix} \text{ and } A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Solution.

$$A = \begin{bmatrix} 3 & 1 \\ -7 & 0 \end{bmatrix}$$

4. **Linear Transformations in Geometry.** [20pts.]

(a) [4pts. each] State whether each matrix in below is an orthogonal projection, reflection, rotation, horizontal shear, or vertical shear.

i. Vertical shear

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

ii. Rotation

$$\begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix}$$

(b) [6pts. each] Let L be the line $y = -x$.

i. Compute the orthogonal projection of the vector $\overrightarrow{\mathbf{v}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ onto L.

ii. Compute the reflection of the vector $\overrightarrow{\mathbf{v}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ about L.

Solution. Let $\overrightarrow{\mathbf{w}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ on L. It is not a unit vector since its length is

$$\|\overrightarrow{\mathbf{w}}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

However, the following is a unit vector on L:

$$\overrightarrow{\mathbf{u}} = \frac{\overrightarrow{\mathbf{w}}}{\|\overrightarrow{\mathbf{w}}\|} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

(a)

$$proj_L(\overrightarrow{\mathbf{v}}) = (\overrightarrow{\mathbf{v}} \cdot \overrightarrow{\mathbf{u}})\overrightarrow{\mathbf{u}} = \left(1\left(\frac{1}{\sqrt{2}}\right) - 0\left(\frac{-1}{\sqrt{2}}\right)\right)\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

(b)

$$ref_L(\overrightarrow{\mathbf{v}}) = 2proj_L(\overrightarrow{\mathbf{v}}) - \overrightarrow{\mathbf{v}} = 2\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

5. **Matrix Products.** [15pts.]

(a) [4pts.] Compute the matrix product if it is defined. If undefined, explain why.

$$\begin{bmatrix} 4 & -1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Solution. The matrix product is undefined since the first matrix is 2×3 and the second is 2×1. The number of columns of the first matrix must equal the number of rows of the second matrix.

(b) [4pts.] Compute the matrix product if it is defined. If undefined, explain why.

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & -1 \end{bmatrix}$$

Solution.

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & -1 \end{bmatrix} = \begin{bmatrix} 1(4) & 1(-1) \\ 3(4) & 3(-1) \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$$

(c) [7pts.] Suppose

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$$

$$L \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ 3x_2 \end{bmatrix}$$

Compute the matrix of the composition $L \circ T$.

Solution. If the matrix of T is A and the matrix of L is B, then the matrix of $L \circ T$ is

$$BA = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left[B \begin{bmatrix} 1 \\ -1 \end{bmatrix} \; B \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix}$$

Scrap Paper