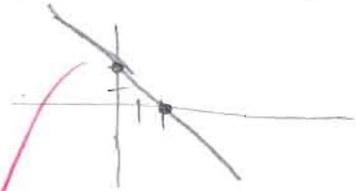
Q1] **[35 points]** There are seven short-answer questions over the next two pages, labeled from a) to g). Each one is worth five points, with a correct answer sufficient for full credit.

a) A system of linear equations in the two unknowns x and y has two solutions: (2, 0) and (0, 2). Find another solution to this system.

$\boxed{(1,1)}$



b) A system of linear equations is represented by the following augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & 4 \\ 0 & -1 & -2 & -3 & 4.5 \end{array}\right] \left[\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & -2 & -3 & -4 \\ 0 & 0 & 0 & 0 & .5 \end{array}\right]$$

inconsistent

How many solutions does it have?

$\boxed{\text{None}}$

c) True/False: Suppose A is a matrix and $\mathbf{v}$ is a vector. Then the product $A\mathbf{v}$ DEFINITELY exists.

$\boxed{\text{False}}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

d) True/False: Suppose T is a linear transformation and $\mathbf{v}$ is a zero vector in the domain of T. Then $T\mathbf{v}$ is also a zero vector.

$\boxed{\text{True}}$

e) What type of transformation (scaling, rotation, reflection, horizontal shear, or vertical shear) is the linear transformation represented by the matrix below? (HINT: $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$.)

$$\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}$$

$\boxed{\text{Rotation}}$

Q1 (continued):

f) Compute the following product of two matrices:

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{I_3} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \pi & e & \pi e & e^{\pi} & \pi^{e} \\ -7 & -6 & 0 & 1 & 8 \end{pmatrix}$$

$$\boxed{\begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ \pi & e & \pi e & e^{\pi} & \pi^{e} \\ -7 & -6 & 0 & 1 & 8 \end{bmatrix}}$$

g) True/False: Suppose A is a 3×3 matrix such that NONE of its nine entries are zeroes. Then A is DEFINITELY invertible.

$\boxed{\text{False}}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

Q2] [20 points]

a) (15 points): Use the Gauss-Jordan method to find the complete solution to the following system of equations:

$$\begin{aligned} w + 2x + 3y + 4z &= 10 \\ -w \qquad + y - 6z &= -6 \\ w + x - y - z &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ -1 & 0 & 1 & -6 & -6 \\ 1 & 1 & -1 & -1 & 0 \end{array}\right] \begin{array}{c} \\ +\mathrm{I} \\ -\mathrm{I} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 2 & 4 & -2 & 4 \\ 0 & -1 & -4 & -5 & -10 \end{array}\right] \begin{array}{c} \\ \div 2 \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & -1 & -4 & -5 & -10 \end{array}\right] \begin{array}{c} -2\mathrm{II} \\ \\ +\mathrm{II} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 6 & 6 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & -2 & -6 & 8 \end{array}\right] \begin{array}{c} \\ \\ \div -2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 6 & \cancel{6} \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & 3 & -4 \end{array}\right] \begin{array}{c} +\mathrm{III} \\ -2\mathrm{III} \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 9 & 2 \\ 0 & 1 & 0 & -7 & 10 \\ 0 & 0 & 1 & 3 & -4 \end{array}\right]$$

$z = t$

$w + 9z = 2$

$x - 7z = 10$

$y + 3z = -4$

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 - 9t \\ 10 + 7t \\ -4 - 3t \\ t \end{bmatrix}}$$

1

-1.

14.

b) (5 points): Using your work from part a), give the reduced row echelon form of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & -6 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 3 \end{bmatrix}}$$

5.

Q3] [20 points]
T is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that

$$T\left(\begin{bmatrix} 7 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 19 \\ 6 \end{bmatrix} \text{ and } T\left(\begin{bmatrix} 6 \\ 6 \end{bmatrix}\right) = \begin{bmatrix} 18 \\ 6 \end{bmatrix}.$$

a) (5 points) Compute

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

(Hint: Subtraction may be very useful here.)

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = T\begin{bmatrix} 7 \\ 6 \end{bmatrix} - T\begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 19 \\ 6 \end{bmatrix} - \begin{bmatrix} 18 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

b) (5 points) Compute

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

from part c

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+2 \\ 0+1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

c) (5 points) Let A be the matrix representing T with respect to the standard basis. What is A?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 7a+6b \\ 7c+6d \end{bmatrix} = \begin{bmatrix} 19 \\ 6 \end{bmatrix}$$

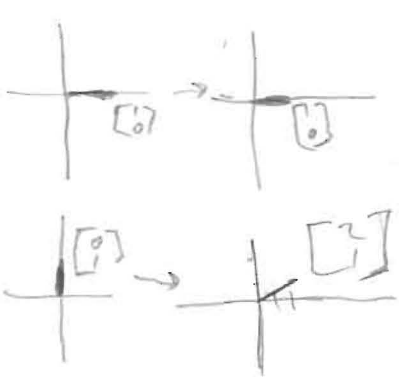
$$\begin{bmatrix} 6a+6b \\ 6c+6d \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 7 & 6 & 0 & 0 & 19 \\ 0 & 0 & 7 & 6 & 6 \\ 6 & 6 & 0 & 0 & 18 \\ 0 & 0 & 6 & 6 & 6 \end{array}\right] \begin{array}{l} -\mathrm{III} \\ -\mathrm{IV} \\ \div 6 \\ \div 6 \end{array} = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} \\ \\ -\mathrm{I} \\ -\mathrm{II} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$a = 1$, $b = 2$, $c = 0$, $d = 1$

$$\boxed{\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}}$$

d) (5 points) What type of transformation (scaling, rotation, reflection, horizontal shear, or vertical shear) is T?



$\boxed{\text{Horizontal Shear}}$

Q4] [25 points]

For this question, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

a) (10 points) Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -2II \\ -2III \\ \end{array} = \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} +III \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\boxed{A^{-1} \cdot \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}}$$

b) (5 points) If $\mathbf{w} = A\mathbf{v}$, compute $\mathbf{w}$.

$$\vec{w} = A\vec{v} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-2+3 \\ 0-1+2 \\ 0\ 0\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}$$

c) (5 points) If $\mathbf{u} = A^{-1}\mathbf{w}$, compute $\mathbf{u}$.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2-2+1 \\ 0+1-2 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

d) (5 points) Using the definitions of $\mathbf{u}$ and $\mathbf{w}$, compute $\mathbf{u}$ in terms of $\mathbf{v}$.

$\vec{u} = A^{-1}\vec{w}$ $\qquad$ $\vec{w} = A\vec{v}$

$\vec{u} = A^{-1}(A\vec{v})$

$\vec{u} = (A^{-1}A)\vec{v}$ $\qquad$ $\vec{u} = I_3\vec{v}$ $\qquad$ $\vec{u} = \vec{v}$