MATH 33A Midterm II, Spring 2018

Name:

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Instruction: Justify all your answers. No points will be given without sufficient reasoning/calculations.

Problem 1. (4)

(i) Use row or column reduction to compute $detA$, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 4 & 5 & 7 \end{bmatrix}.$$

(ii) Find $detB$, where

$$B = \begin{bmatrix} 0 & 0 & a & 4 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & b & c \\ 1 & d & e & f \end{bmatrix}.$$

Here a, b, c, d, e and f are constants.

(iii) Find $det(AB)$, $det(A^{-1})$ and $det(B^{-1})$.

(i) $$|A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 4 \\ 0 & 3 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -2 & -3 \end{vmatrix} = \boxed{-1}$$

1.5

(ii) $$|B| = \begin{vmatrix} 0 & 2 & b & c \\ 1 & d & e & f \\ 0 & 0 & a & 4 \\ 0 & 0 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 1 & d \end{vmatrix} \cdot \begin{vmatrix} a & 4 \\ 3 & 0 \end{vmatrix}$$

1.5

$$= (-2) \cdot (-12) = \boxed{24}$$

(iii) $$|AB| = |A|\,|B| = \boxed{-24} \qquad |A^{-1}| = |A|^{-1} = \boxed{-1}$$

1

$$|B^{-1}| = |B|^{-1} = \boxed{\frac{1}{24}}$$

Problem 2. (4)

Let V =span $\{\vec{v_1}, \vec{v_2}\}$ be a subspace in $\mathbf{R}^3$, where $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(i) Find an orthonormal basis for V by using Gram-Schmidt process.

(ii) Find the QR-factorization for A, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

(i) $\boxed{w_1} = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$V_2 \cdot w_1$ $\qquad$ $V_2 \cdot w_1$

$\tilde{w}_2 = V_2 - \langle V_2, w_1 \rangle w_1$, $\qquad \langle V_2, w_1 \rangle = \frac{1}{\sqrt{3}}$

$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

2 pts

$= \frac{1}{3}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$, $\quad \boxed{w_2} = \frac{\tilde{w}_2}{\|\tilde{w}_2\|} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

(ii) $A = (v_1, v_2) = QR$

$= (w_1 \ w_2)\begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix}$

$r_{11} = v_1 \cdot w_1 = \sqrt{3}$

$r_{12} = v_2 \cdot w_1 = \frac{1}{\sqrt{3}}$

$r_{22} = v_2 \cdot w_2 = \frac{2}{\sqrt{6}}$

2 pts

$\therefore Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \end{pmatrix} \qquad R = \begin{pmatrix} \sqrt{3} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} \end{pmatrix}$

Problem 3. (4)

(i) Decide if the equation $A\overrightarrow{x} = \overrightarrow{b}$ solvable, where $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$, and $\overrightarrow{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(ii) Find the least-squares solution of the equation $A\overrightarrow{x} = \overrightarrow{b}$ for A and $\overrightarrow{b}$ in (i).

(i) $A\vec{x} = \vec{b}$ $(*)$ is solvable $\Leftrightarrow$ $rk(\tilde{A}) = rk(A)$

$$\tilde{A} = \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right) \quad rk(\tilde{A}) = 3 > 2 = rk(A)$$

2 pts $\Longrightarrow$ $(*)$ is not solvable.

(ii) Normal equation for least square solution:

2 pts $$A^*A X = A^*b = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\|$$

3

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} X = \overset{A^*A}{\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}} X$$

$$X = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\Big/\underbrace{2}_{\det A^*A} = \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\Big/2}$$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}\Big/2$

Since $proj_{V^\perp}(\vec{X}) = W(W \underset{dot}{\cdot} \vec{X}) = (W \cdot W^T)\vec{X}$.

Problem 4. (4)

Let $V = \text{span}\{\vec{v_1}, \vec{v_2}\}$, where $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{v_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(i) Find a basis for the orthogonal complement $V^\perp$ of V.

(ii) Find the matrix for the projection to $V^\perp$, $Proj_{V^\perp}$: $\mathbb{R}^3 \to \mathbb{R}^3$ w.r.t the std basis.

(i) Since $\dim V = 2 \Rightarrow \dim V^\perp = 3 - 2 = 1$

Let $V^\perp = \text{span}\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right\} \Rightarrow$

$$\begin{matrix} V_1 \underset{dot}{\cdot} \vec{X} = 0 \\ V_2 \underset{dot}{\cdot} \vec{X} = 0 \end{matrix} \Longleftrightarrow \begin{pmatrix} V_1^T \cdot X \\ V_2^T \cdot X \end{pmatrix} = \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} X$$

$$= (V_1 \; V_2)^T X = 0$$

2 pts

$$\Updownarrow$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Longrightarrow$$

$$\vec{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}} \qquad \text{Let } W = \frac{\vec{X}}{\|X\|} = \boxed{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \Big/ \sqrt{2}}$$

be the orth'l basis for $V^\perp$.

$\Rightarrow$ (ii) The matrix $= W \cdot W^T = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}(-1, 1, 0)$

2 pts $\quad = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$E_{\lambda_2} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$

0.5 (iii) since for λ_1, alg. mult $= 2 >$ geo. mult. $= 1$

A is not diagonalizable.

Problem 5. (4)

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$

(i) Find the eigenvalues of A and their algebraic multiplicities.

(ii) Find a basis for each eigenspace of A, and find the geometric multiplicity for each eigenvalue of A.

(iii) Decide if A is diagonalizable.

(i) $f_A(\lambda) = \det(A - \lambda I) = \det\begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & -2 \\ 0 & 1 & 4-\lambda \end{bmatrix}$

$= (2-\lambda)\det\begin{pmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{pmatrix} = (2-\lambda)(\lambda^2 - 5\lambda + 6)$

$= (2-\lambda)(\lambda-2)(\lambda-3) = 0 \Rightarrow$

1.5

$\lambda_1 = 2$ w/ alg. mult $= 2$

$\lambda_2 = 3$ w/ ——— $= 1$.

(ii) E_{λ_1} : $(A - \lambda_1 I)X = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\Downarrow$

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Updownarrow$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \Rightarrow \vec{X} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\text{rk}(A - \lambda_1 I) = 2$

$\Rightarrow \dim(E_{\lambda_1}) = 1$

$\shortparallel$

2 pb geometric mult

& $E_{\lambda_1} = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

E_{λ_2} : $(A - \lambda_2 I)X = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$X = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ since $\text{rk}(A - \lambda_2 I) = 2 \Rightarrow \dim(E_{\lambda_2}) = 1$