Math 33A Midterm 1, October 19, 2012

Name: ______________________

Student ID. ______________________

Instructions: Show all of your work, and clearly indicate your answers. Use the backs of pages as scratch paper. No books, other paper, or calculators are allowed.

1. (15 points) The reduced row echelon form of the augmented matrices of three systems are given below. How many solutions does each system have?

(a) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$;

(b) $\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 6 \end{array}\right]$;

(c) $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$.

(a) Because the last row ~~of~~ the matrix is in the form $[0\,0\,0 \vdots 1]$, we can know that there is $\boxed{\text{no solution}}$ for this system.

(b) The system matrix A of this system is a 2×2 matrix, and $\operatorname{rank}(A) = 2$. So ~~the~~ it is a consistent system and has $\boxed{\text{a unique solution}}$

(c) The system matrix B of this system is a 2×2 matrix and $\operatorname{rank}(B) = 2$. So ~~it~~ has either infinitely many solutions or none. There is no row in the form $[0\,0\,0 \vdots n]$ $n \neq 0$

Therefore, there are $\boxed{\text{infinitely many solutions}}$

2. (15 points) Find all matrices that commute with the given matrix A,

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Let B be the matrix that commutes with A.

$AB = BA$ implies that B is also a 3×3 matrix.

Suppose $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} 2b_{11} & 2b_{12} & 2b_{13} \\ 2b_{21} & 2b_{22} & 2b_{23} \\ 3b_{31} & 3b_{32} & 3b_{33} \end{bmatrix}$$

$$BA = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2b_{11} & 2b_{12} & 3b_{13} \\ 2b_{21} & 2b_{22} & 3b_{23} \\ 2b_{31} & 2b_{32} & 3b_{33} \end{bmatrix}$$

$AB = BA$.

$$\begin{cases} 2b_{11} = 2b_{11} \\ 2b_{12} = 2b_{12} \\ 2b_{13} = 3b_{13} \\ 2b_{21} = 2b_{21} \\ 2b_{22} = 2b_{22} \\ 2b_{23} = 3b_{23} \\ 3b_{31} = 2b_{31} \\ 3b_{32} = 2b_{32} \\ 3b_{33} = 3b_{33} \end{cases} \Rightarrow \begin{cases} b_{13} = 0 \\ b_{23} = 0 \\ b_{31} = 0 \\ b_{32} = 0 \end{cases}$$

All matrices that satisfy $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix}$ where a,b,c,d,e are arbitrary numbers commute with A. and $a,b,c,d,e \in \mathbb{R}$

3. (15 points) Which of the following are linear transformations? Write down the matrix of the transformation if a function is linear.

(a) $T(a, b) = [a^2, b]$;
(b) $T(a) = [a, 2^a]$;
(c) $T(a, b) = [3a - b, -2a + 3b]$;
(d) $T(a, b) = [b + a, a + 1]$.

(a) $T(ka, kb) = [k^2a^2, kb] \neq kT(a,b)$ $\quad k \in \mathbb{R}$

(a) is not a linear transformation.

(b) $T(ka) = [ka, 2^{ka}] \neq kT(a)$ $\quad k \in \mathbb{R}$

(b) is not a linear transformation.

(c) $T(ka, kb) = [3ka - kb, -2ka + 3kb] = kT(a,b)$ $\quad k \in \mathbb{R}$

$T(a+c, b+d) = [3(a+c) - (b+d), -2(a+c) + 3(b+d)] = T(a,b) + T(c,d)$ $\quad c, d \in \mathbb{R}$

(c) is a linear transformation.

The matrix of this transformation is $\begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix}$

(d) $T(ka, kb) = [kb + ka, ka + 1] \neq k(T(a,b))$ $\quad k \in \mathbb{R}$

(d) is not a linear transformation.

4. (15 points) Suppose A is an $m \times n$ matrix. If the kernel of A, $\{\vec{x} \mid A\vec{x} = \vec{0}\}$, does not just have the 0 vector, show that the rank of $A < n$.

$A\vec{x} = \vec{0}$ can be written as

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{m \times 0}$$

The augmented matrix for this equation is

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} & 0 \\ a_{21} & \cdots & a_{2m} & 0 \\ \vdots & & & \\ a_{n1} & \cdots & a_{nm} & 0 \end{bmatrix}$$

because the last column contains all zero, no matter what kind of row reduction we use, it remains the same.

Since the solution of this equation does not just have 0 vector which means there is at least one row is in the form $[\overbrace{0\,0 \ldots}^{m}\, 0\, 0]$ or we can say there is free variable exists. Thus the reduced row electron form has at least one row full of zero.

Therefore $\mathrm{rank}(A) < n$

5. (20 points) Consider the following linear system:

$$\begin{bmatrix} 1 & 1 & k \\ 1 & 2 & 2k \\ 2 & 2 & 3k \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}.$$

(a) Find the condition for k, such that this linear system has a unique solution.
(b) What is the unique solution if given the condition for k as you describe in (a).

(a) The augmented matrix A of this system is

$$A = \begin{bmatrix} 1 & 1 & k & 1 \\ 1 & 2 & 2k & 3 \\ 2 & 2 & 3k & 6 \end{bmatrix} \begin{matrix} \to \\ -R_1 \\ -2R_1 \end{matrix} \begin{bmatrix} 1 & 1 & k & 1 \\ 0 & 1 & k & 2 \\ 0 & 0 & k & 4 \end{bmatrix} \begin{matrix} -R_2 \\ -R_3 \\ \to \end{matrix} \begin{bmatrix} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & k & 4 \end{bmatrix} \begin{matrix} -R_2 \\ \to \\ \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & k & 4 \end{bmatrix}$$

For this system has a unique solution, $k \neq 0$

(b) if $k \neq 0$, the solution of this system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ \frac{4}{k} \end{bmatrix}$$

6. (20 points) Consider the following 3×4 matrix:

$$A = \begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 0 & 2 & 2 \\ 3 & 1 & 7 & 8 \end{bmatrix}$$

(a) Find Ker(A), describe it as the span of some vectors;
(b) Find a matrix B, such that Im(B)=Ker(A). Explain your answer.

(a) Let $\ker(A) = \vec{x}$ ~~such th~~

$$A\vec{x} = 0$$

$$\begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 0 & 2 & 2 \\ 3 & 1 & 7 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$$

Solve this equation.

$$\begin{bmatrix} 1 & 1 & 3 & 4 & 0 \\ 1 & 0 & 2 & 2 & 0 \\ 3 & 1 & 7 & 8 & 0 \end{bmatrix} \xrightarrow{\leftrightarrow R_1} \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 1 & 1 & 3 & 4 & 0 \\ 3 & 1 & 7 & 8 & 0 \end{bmatrix} \begin{matrix} \\ -R_1 \\ -3R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \begin{matrix} \\ \\ -R_2 \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} x_1 + 2x_3 + 2x_4 = 0 \\ x_2 + x_3 + 2x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2x_3 - 2x_4 \\ x_2 = -x_3 - 2x_4 \end{cases}$$

Let $x_3 = s$ $x_4 = t$. $s, t \in \mathbb{R}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s-2t \\ -s-2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Ker(A) is span of vectors $\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

(b) $$Im(B) = s\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Let $B = [\vec{v}_1 \ \vec{v}_2]$ $\vec{v}_1$ and $\vec{v}_2$ are column vectors

$$[\vec{v}_1 \ \vec{v}_2]\vec{x} = v_1x_1 + v_2x_2 = s\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

~~$= h + v$~~

One solution is

$$\vec{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & -2 \\ -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$