Name: [redacted] Student ID: [redacted] Section: 2F

1 30
2 21
3 40
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Math 33A/2 – MIDTERM EXAMINATION
Spring 2012

Instructions:

(a) The exam is closed-book (except for one page of notes) and will last 50 minutes. No calculators, cell phones, or other electronic devices are allowed at any time.

(b) Notation will conform as closely as possible to the standard notation used in the lectures, not the textbook.

(c) Do all 3 problems. Problem 1 and 2 are worth 30 points, problem 3 is worth 40 points. Hand in the exam with work shown where appropriate. Some partial credit may be assigned if warranted. Label clearly the problem number and the material you wish to be graded.

(d) You must enter your discussion section above. Failure to provide the correct discussion section will result in a loss of 5 points.

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1. 10 (a) Consider the (symmetric) matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$. Perform Gaussian elimination on A to reduce it to upper triangular form U.

10 (b) Write A in the form LU where L is unit lower triangular, i.e., find L explicitly.

10 (c) Compute $\det(A)$ directly from the LU factorization of A?

a 1-b. $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -6 \\ 4 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -6 \\ 0 & -6 & -15 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -6 \\ 0 & 0 & -3 \end{bmatrix}$ ✓

$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 4 \\ 0 & -3 & -6 \\ 0 & 0 & -3 \end{bmatrix}}_{U} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$ ✓

c $\det(A) = 1 \cdot (-3)(-3) = 9$

$1(1-4) - 2(2-8) + 4(4-4)$
$-3 + 12 = 9$

3. Let

$$A = \begin{bmatrix} 4 & 2 & 3 & 0 \\ 2 & 1 & 2 & 1 \\ 6 & 3 & 5 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 4}.$$

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(a) Put A in reduced row echelon form (rref) R.

(b) Determine rank(A) and state why you have given the answer you have given. Determine rank(A^T) and state why you have given the answer you have given.

(c) Find a basis for $\mathcal{N}(A)$ (the nullspace of A) from $R = \text{rref}(A)$.

(d) Find a basis for $\mathcal{C}(A) = \mathcal{R}(A)$ (the columnspace of A = the range of A) from $R = \text{rref}(A)$.

(e) Find a basis for $\mathcal{C}(A^T) = \mathcal{R}(A^T)$ from $R = \text{rref}(A)$.

(f) The matrix $E = \begin{bmatrix} 1 & -\frac{3}{2} & 0 \\ -1 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ puts A in rref. Using this E, find a basis for $\mathcal{N}(A^T)$.

a. $\begin{bmatrix} 4 & 2 & 3 & 0 \\ 0 & 0 & 1/2 & 1 \\ 0 & 0 & 1/2 & 1 \end{bmatrix} \to \begin{bmatrix} 4 & 2 & 3 & 0 \\ 0 & 0 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 3/4 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 0 & -3/2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \text{rref}(A)$

b. rank$(A) = 2$ because there are 2 pivots in rref(A)
rank$(A^T) = 2$ because rank$(A) = 2$ also

c. $x_1 + \frac{x_2}{2} - \frac{3x_4}{2} = 0$ when $x_4 = 1$ $x_2 = 0$ when $x_2 = 1$ $x_4 = 0$
$x_3 + 2x_4 = 0$ $x_1 = \frac{3}{2}$, $x_3 = -2$ $x_1 = -\frac{1}{2}$ $x_3 = 0$

$\begin{bmatrix} 3/2 \\ 0 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

d. $\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$

e. $\begin{bmatrix} 1 \\ 1/2 \\ 0 \\ -3/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

f. $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

2. Write all solutions of the equations $Ax = b$ where $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ in the form $x_p + x_n$ where x_p denotes the particular solution which solves $Ax_p = b$ and the special solutions x_n solve $Ax_n = 0$.

$= \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -7 \end{array}\right] \to$ $x_1 + 2x_2 + 3x_3 = 4$

$4x_2 - 8x_3 = -7$

$x_1 + (-\frac{1}{2}) + 3 = 4$

$x_1 + \frac{5}{2} = \frac{8}{2}$

$x_1 = \frac{3}{2}$

$x_p = \begin{bmatrix} 3/2 \\ -1/4 \\ 1 \end{bmatrix}$ actually $x_3 = 0$

$-4x_2 = 5 + 8$

$x_1 - 6 + 3 = 4$

$x_2 = -3$

$x_1 = 7$

x_n: $x_1 + 2x_2 + 3x_3 = 0$

$-x_2 - 2x_3 = 0$

$x_1 - 4 + 3 = 0$

$x_2 + 2x_3 = \frac{7}{4}$

$4x_2 + 8x_3 = 7$

$x_n = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$x = x_p + x_n = \begin{bmatrix} 3/2 \\ -1/4 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

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nice! good

check: $x_1 + 2x_2 + 3x_3 \stackrel{?}{=} 4$

$3 + (-2) + 3 = 4$ ✓

$2 + -6 +$

$\frac{9}{2} - \frac{1}{2} + 1 = 5$ ✓

$3x_1 + 2x_2 + x_3 = 5$

$3 \cdot \frac{13}{2} - \frac{1}{2} + 1 \stackrel{?}{=} 5$

$9 - 2 + 1 = 5$

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