Math 33A
Spring 2017
Exam 2
5/24/17
Time Limit: 50 Minutes

This exam contains 6 pages (including this cover [redacted] any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	12	10
2	10	8
3	10	8
4	10	10
5	8	8
Total:	50	44

1. Consider the matrix

10/12

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix}$$
$\vec{v}_1 \quad \vec{v}_2$

(a) (6 points) Find the QR factorization of A.

6/6

$\vec{q}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{5}\begin{bmatrix}3\\0\\4\end{bmatrix} = \begin{bmatrix}3/5\\0\\4/5\end{bmatrix}$

$\vec{q}_2 = \frac{\vec{v}_2 - (\vec{v}_2\cdot\vec{q}_1)\vec{q}_1}{\|\vec{v}_2 - (\vec{v}_2\cdot\vec{q}_1)\vec{q}_1\|} = \frac{1}{\|\ldots\|}\left(\begin{bmatrix}2\\1\\1\end{bmatrix} - \left(\underbrace{\begin{bmatrix}2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}3/5\\0\\4/5\end{bmatrix}}_{\frac{6}{5}+\frac{4}{5}=\frac{10}{5}=2}\right)\begin{bmatrix}3/5\\0\\4/5\end{bmatrix}\right) = \frac{1}{\|\ldots\|}\left(\begin{bmatrix}2\\1\\1\end{bmatrix} - 2\begin{bmatrix}3/5\\0\\4/5\end{bmatrix}\right)$

$= \frac{1}{\|\ \|}\begin{bmatrix}2-6/5\\1-0\\1-8/5\end{bmatrix} = \frac{1}{\|\ \|}\begin{bmatrix}10/5-6/5\\1\\5/5-8/5\end{bmatrix} = \frac{1}{\|\ \|}\begin{bmatrix}4/5\\1\\-3/5\end{bmatrix}$

$\|\ldots\| = \sqrt{\frac{16}{25}+1+\frac{9}{25}} = \sqrt{2}$

$\vec{q}_2 = \begin{bmatrix}4/5\\1\\-3/5\end{bmatrix}\frac{1}{\sqrt{2}}$

$Q = \begin{bmatrix} | & | \\ \vec{q}_1 & \vec{q}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 3/5 & 4/5\sqrt{2} \\ 0 & 1/\sqrt{2} \\ 4/5 & -3/5\sqrt{2} \end{bmatrix}$

$R = \begin{bmatrix} \vec{q}_1\cdot\vec{v}_1 & \vec{q}_1\cdot\vec{v}_2 \\ 0 & \vec{q}_2\cdot\vec{v}_2 \end{bmatrix} = \begin{bmatrix} 9/5+16/5 & 2 \\ 0 & 8/5\sqrt{2}+1/\sqrt{2}-3/5\sqrt{2} \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 0 & 2/\sqrt{2} \end{bmatrix}$

$\hookrightarrow \frac{5}{5\sqrt{2}}+\frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}}$

$\boxed{A = \begin{bmatrix} \frac{3}{5} & \frac{4}{5\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \\ \frac{4}{5} & \frac{-3}{5\sqrt{2}} \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 0 & 2/\sqrt{2} \end{bmatrix}}$

check: $\begin{bmatrix} 3/5 & \frac{4}{5\sqrt{2}} \\ 0 & 1/\sqrt{2} \\ 4/5 & \frac{-3}{5\sqrt{2}} \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 0 & 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 & 6/5+8\sqrt{2}/5\sqrt{2} \\ 0 & 1 \\ 4 & 8/5-\frac{6\sqrt{2}}{5\sqrt{2}} \end{bmatrix}$ $\frac{14}{5}$, $\frac{2}{5}$?? ~~doesn't check out..~~

(b) (6 points) Find the vector $\hat{\mathbf{x}}$ such that the distance between $A\hat{\mathbf{x}}$ and $\mathbf{b} = \begin{bmatrix}0\\1\\0\end{bmatrix}$ is minimal.

4/6

(Hint: it might help to use part (a)).

$A\hat{x} = \vec{b}$

$R\hat{x} = Q^T\vec{b}$

$\hat{x} = R^{-1}Q^T\vec{b}$

oops

Find R^{-1}:

$\begin{bmatrix} 5 & 2 \\ 0 & 2/\sqrt{2} \end{bmatrix}$ $R^{-1} = \frac{1}{10/\sqrt{2}}\begin{bmatrix} 2/\sqrt{2} & 0 \\ -2 & 5 \end{bmatrix}$

Find Q^T:

$\begin{bmatrix} 3/5 & 0 & 4/5 \\ \frac{4}{5\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{-3}{5\sqrt{2}} \end{bmatrix}$

X

$R^{-1}Q^T = \begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{1}{5\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{bmatrix}\begin{bmatrix} 3/5 & 0 & 4/5 \\ \frac{4}{5\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{3}{5\sqrt{2}} \end{bmatrix}$

$= \begin{bmatrix} \frac{3}{25} & 0 & \frac{4}{25} \\ -\frac{3}{25\sqrt{2}}+\frac{1}{5} & \frac{1}{4} & -\frac{4}{25\sqrt{2}}-\frac{3}{20} \end{bmatrix}$

$R^{-1}Q^T\vec{b} = \begin{bmatrix}0\\\frac{1}{4}\end{bmatrix}$

$\boxed{\hat{x} = \begin{bmatrix}0\\\frac{1}{4}\end{bmatrix}}$

$\frac{8}{10}$

2. Consider the ordered basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$. (You don't need to verify that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.)

(a) (6 points) Find the coordinates of the vector

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$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

with respect to the basis $\mathcal{B}$.

$[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}$ $\quad S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}$

Find S^{-1}: $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & -1 & -3 & -1 & 1 & 0 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 0 & 6 & 1 & -2 & 1 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 1/6 & -1/3 & 1/6 \end{array}\right]$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/6 & 1/3 & 1/6 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & 1/6 & -1/3 & 1/6 \end{array}\right]$ S^{-1}

$\begin{array}{ccc} 1/3 & & 1/3 \end{array}$

$S^{-1}\vec{x} = \begin{bmatrix} 1/6 & 1/3 & 1/6 \\ 1/2 & 0 & -1/2 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\boxed{[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$

(b) (4 points) Suppose that

3/4

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

is the matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the standard basis. Find the matrix of T with respect to $\mathcal{B}$. You don't have to evaluate the matrix products and/or inverses that appear in your formula.

matrix of T wrt $\mathcal{B}$ = B

B ~ A thus $B = S^{-1}AS$

$$\boxed{B = \begin{bmatrix} 1/6 & 1/3 & 1/6 \\ 1/2 & 0 & -1/2 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}}$$

3. (10 points) Indicate whether the following statements are true or false by circling the appropriate answer. You don't need to justify your answer, and there is no partial credit.

$\det\left(\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}-\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\right)$

TRUE / FALSE: A 3×3 matrix with real entries always has a real eigenvalue.

not necessarily

$= \begin{vmatrix}(\lambda-1)&-2&-3\\-4&(\lambda-5)&-6\\-7&-8&(\lambda-9)\end{vmatrix}$

$\det = (\lambda-1)\begin{vmatrix}\lambda-5&-6\\-8&\lambda-9\end{vmatrix} + 2\begin{vmatrix}-4&-6\\-7&\lambda-9\end{vmatrix} - 3\begin{vmatrix}-4&\lambda-5\\-7&-8\end{vmatrix}$

TRUE / FALSE: If A is the matrix

not possible! $\det(B^2) = \det(B)\cdot\det(B) = -1$? $\begin{bmatrix}1&1\\2&1\end{bmatrix}$ $\det(A) = -1$

then there is a 2×2 matrix B with real entries such that $B^2 = A$.

$B^2 = \begin{bmatrix}b_1&b_2\\b_3&b_4\end{bmatrix}\begin{bmatrix}b_1&b_2\\b_3&b_4\end{bmatrix} = \begin{bmatrix}b_1^2+b_2b_3 & b_1b_2+b_2b_4\\ b_1b_3+b_3b_4 & b_3b_2+b_4^2\end{bmatrix} \cdot \begin{bmatrix}1&1\\2&1\end{bmatrix}$

TRUE / FALSE: If V is a subspace of $\mathbb{R}^n$, then $\dim V + \dim V^{\perp} = n$.

TRUE / FALSE: If the columns of a square matrix A are orthogonal, then A is an orthogonal matrix.

$Q^TQ = I_n$
$Q^T = Q^{-1}$

$A = \begin{bmatrix}| & | \\ \vec{a}_1 & \vec{a}_2 \\ | & |\end{bmatrix}$ $A^{-1} = \begin{bmatrix}- \vec{x}_1 - \\ - \vec{x}_2 -\end{bmatrix}$

$A^T = \begin{bmatrix}- \vec{a}_1 - \\ - \vec{a}_2 -\end{bmatrix}$ where $\vec{a}_1 \cdot \vec{x}_1 = 1$, $\vec{a}_2 \cdot \vec{x}_2 = 0$

$A^TA = \begin{bmatrix}\vec{a}_1\cdot\vec{a}_1 \\ \vec{a}_2\cdot\vec{a}_2\end{bmatrix}$ $x_1 = a_1$, $x_2 = a_2$?

$\vec{a}_1 \cdot \vec{a}_2 = 0$

b/c $A^T = A^{-1}$

not necessarily b/c if $\vec{x}_2 = \vec{a}_2$, $\|\vec{a}_2\| = 0$ ... a_2 can be anything

TRUE / FALSE: The matrices

$$A = \begin{bmatrix}3&4&-1\\2&2&2\\0&1&1\end{bmatrix} \quad B = \begin{bmatrix}2&1&0\\3&2&1\\5&4&3\end{bmatrix}$$

are similar.

$tr(A) = 6 \neq tr(B) = 7$

4. (a) (5 points) Suppose that A is a 2×2 matrix with real entries. Verify that

5 $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$

$$\det(A) = \frac{\operatorname{tr}(A)^2 - \operatorname{tr}(A^2)}{2}$$

$tr(A) = a_1 + a_4$

$tr(A)^2 = a_1^2 + a_4^2 + 2a_1a_4$

$A^2 = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} a_1^2 + a_2a_3 & a_1a_2 + a_2a_4 \\ a_3a_1 + a_3a_4 & a_2a_3 + a_4^2 \end{bmatrix}$

$tr(A^2) = a_1^2 + 2a_2a_3 + a_4^2$ ✓

$\dfrac{tr(A)^2 - tr(A^2)}{2} = \frac{1}{2}\left(a_1^2 + a_4^2 + 2a_1a_4 - a_1^2 - 2a_2a_3 - a_4^2\right)$ ✓

$= \frac{1}{2}(2a_1a_4 - 2a_2a_3) = a_1a_4 - a_2a_3 = det(A)$ ∎

(b) (5 points) Let A be a 2×2 matrix with real entries such that

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$$\operatorname{tr}(A) = 0 \quad \operatorname{tr}(A^2) = 2$$

What are the eigenvalues of A? (Hint: use part (a).)

$P_A(\lambda) = det(\lambda I_n - A)$

$= \begin{vmatrix} \lambda - a_1 & a_2 \\ a_3 & \lambda - a_4 \end{vmatrix} = (\lambda - a_1)(\lambda - a_4) - a_2a_3 = \lambda^2 + \underbrace{a_1a_4 - a_2a_3}_{det(A)} - \lambda a_1 - \lambda a_4$

$det(A) = \dfrac{tr(A)^2 - tr(A^2)}{2} = \dfrac{0^2 - 2}{2} = -1$

$P_A(\lambda) = \lambda^2 + det(A) - \lambda a_1 - \lambda a_4 = \lambda^2 - 1 - \lambda a_1 - \lambda a_4 = 0$

$\lambda^2 - \underbrace{(a_1 + a_4)}_{tr(A) = 0}\lambda - 1 = 0$

$\lambda^2 - 1 = 0$

$\lambda^2 = 1$

$\boxed{\lambda = \pm 1}$

5. The matrix

$$A = \begin{bmatrix} 2 & 0 & -1 & 5 & 4 \\ 1 & 2 & 3 & 2 & 1 \\ -1 & 1 & 0 & 1 & 1 \\ 3 & 5 & 1 & 2 & 1 \\ 0 & 7 & 0 & 1 & 1 \end{bmatrix}$$

has determinant 22 (you don't need to show this!). Find the determinant of the following matrices. In each case give a brief explanation.

(a) (2 points)

$$B_1 = \begin{bmatrix} 2 & 0 & -1 & 5 & 4 \\ 0 & 7 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 \\ 3 & 5 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 & 1 \end{bmatrix}$$ swap

✓ $\det(B_1) = -\det(A)$ if B_1 is derived from A by interchanging one row. Thus $\boxed{\det(B_1) = -22}$

(b) (3 points)

$$B_2 = \begin{bmatrix} 2 & 1 & -1 & 3 & 0 \\ 0 & 2 & 1 & 5 & 7 \\ -1 & 3 & 0 & 1 & 0 \\ 5 & 2 & 1 & 2 & 1 \\ 4 & 1 & 1 & 1 & 1 \end{bmatrix}$$

✓ $\det(A^T) = \det(A)$ and $B_2 = A^T$ so $\det(B_2) = \det(A)$

$\boxed{\det(B_2) = 22}$

(c) (3 points)

$$B_3 = \begin{bmatrix} 2 & 0 & -1 & 5 & 4 \\ 1 & 2 & 3 & 2 & 1 \\ 0 & 3 & 3 & 3 & 2 \\ 3 & 5 & 1 & 2 & 1 \\ 0 & 7 & 0 & 1 & 1 \end{bmatrix}$$ ← added R_2 to R_3

✓ $\det(B_3) = \det(A)$ if B_3 is derived from A by adding a scalar multiple of one row to another. Thus

$\boxed{\det(B_3) = 22}$

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