

# 19F-MATH33A-3 Midterm I

TOTAL POINTS

**45 / 50**

QUESTION 1

1 12 / 15

- ✓ - 1 pts (b) didn't check  $-3-2k \neq 0$
- ✓ - 2 pts (b) wrong value for k when inconsistent

QUESTION 2

2 15 / 15

- ✓ + 1 pts (a) Said "yes" or "linear"
- ✓ + 2 pts (a) Correct matrix
- ✓ + 4 pts (b) Sensible work to compute inverse
- ✓ + 3 pts (b) Correct inverse
- ✓ + 3 pts (c) Set up some sensible method (row reduction or multiplying by inverse)
- ✓ + 2 pts (c) Correct solution vector (give points if based on incorrect inverse, will deduct 1 below)
  - 1 pts Used incorrect inverse to find (c)

QUESTION 3

3 8 / 10


- ✓ - 2 pts Matrix for T missing

QUESTION 4

4 10 / 10

- ✓ - 0 pts correct answer

## Midterm 1

Last Name:  \_\_\_\_\_First Name:  \_\_\_\_\_Student ID:  \_\_\_\_\_Signature:  \_\_\_\_\_

Section:

Tuesday:

Thursday:

3A

3B

TA: LI, XIA

3C

3D

TA: ANDERSON, AARON

3E

3F

TA: JOHNSON, ALEXANDER

**Instructions:** Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators, books, notes, or any other material to help you.** Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly. I will grade only work on the exam paper, unless you clearly indicate your desire for me to grade work on additional pages.

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Please do not write below this line.



Problem	Max	Score
1	15	
2	15	
3	10	
4	10	
Total	50	



## Summary of 2D Geometrical Linear Transformations

- Scaling:  $kI_2 = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ .
- Projection:  $P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$  where  $a = u_1^2$ ,  $b = u_1u_2$  and  $c = u_2^2 = 1 - u_1^2$  (Unit vector of line  $L: \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ )
- Reflection:  $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ , where  $a = 2u_1^2 - 1$  and  $b = 2u_1u_2$  (Unit vector of line  $L: \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ ).
- Rotation combined with a scaling:  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  (Note: scaling factor  $r = \sqrt{a^2 + b^2}$ , rotation angle  $\theta = \tan^{-1}(b/a)$ ).
- Horizontal shear:  $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ , vertical shear  $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$



1. (15 pts) Consider the linear system

$$\begin{cases} 3x_1 + kx_2 = 6 \\ kx_1 + x_2 = -3 \end{cases} \quad 2x_2 = 9$$

(a) (3 pts) Write the systems in matrix form and write down the corresponding augmented matrix.

$$\begin{bmatrix} 3 & k \\ k & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$\rightarrow \left[ \begin{array}{cc|c} 3 & k & 6 \\ k & 1 & -3 \end{array} \right]$$

(b) (6 pts) Determine the values of  $k$  such that the above linear system is consistent.

$$3x_1 + kx_2 = -2kx_1 - 2x_2 \quad \left[ \begin{array}{cc|c} 1 & k/3 & 2 \\ k & 1 & -3 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & k/3 & 2 \\ 0 & 1 - \frac{k^2}{3} & -3 - 2k \end{array} \right]$$

$(3 - 2k)x_1 = (-2 - k)x_2$   
 $(2k - 3)x_1 = (k + 2)x_2$   
 $x_2 = \frac{k+2}{2k-3} x_1$

$$\left[ \begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array} \right] \quad \left( \frac{k(k+2) + 2k - 3}{2k-3} \right) x_2 = -3$$

$k$  consistent for all  $\mathbb{R}$ ; except  $k = \frac{3}{2}$ ;  $k = 1$

(c) (6 pts) If  $k = 3$ , is the linear system consistent? If it is consistent, please find all the solutions.

$$\left[ \begin{array}{cc|c} 3 & 3 & 6 \\ 3 & 1 & -3 \end{array} \right] \rightarrow \left[ \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -9 \end{array} \right]$$

$$\rightarrow \left[ \begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -9 \end{array} \right]$$

$$\begin{aligned} x_1 + x_2 &= 2 \\ -2x_2 &= -9 \\ x_2 &= \frac{9}{2} \end{aligned}$$

Yes, consistent.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{9}{2} \end{bmatrix}$$





2. (15 pts) Define the transformation  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  so that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} x_1 - 2x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + 3x_2 + 4x_3 \end{bmatrix}$$

(a) (3 pts) Is this transformation linear? If so find its matrix  $A$ .

$$T(\vec{x}) = A\vec{x} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Yes it is linear

$$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$$

$$T(k\vec{x}) = kT(\vec{x})$$

(b) (7 pts) Is  $A$  invertible? If  $A$  is invertible, please find  $A^{-1}$ .

Yes invertible

$$[A | I_3] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array} \right] = \left[ \begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & -1 & 1 & 0 \\ 0 & 5 & 3 & -1 & 0 & 1 \end{array} \right]$$

$$= \left[ \begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 5 & 3 & -1 & 0 & 1 \end{array} \right] = \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1/3 & 2/3 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & 3 & -5/3 & -1 & 1 \end{array} \right] = \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1/3 & 2/3 & 0 \\ 0 & 1 & 0 & -2/3 & 1/3 & 0 \\ 0 & 0 & 1 & -5/3 & -1 & 1 \end{array} \right]$$

(c) (5 pts) If there is any, find a vector  $\vec{v}$  such that  $T(\vec{v}) = \vec{b}$  where

$$\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix}$$

$$T(\vec{v}) = \vec{b} \rightarrow A\vec{v} = \vec{b} \quad A^{-1}\vec{b} = \vec{v}$$

$$\vec{v} = A^{-1}\vec{b} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & -1/3 \\ -5/3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \vec{v}$$

$$A^{-1} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & -1/3 \\ -5/3 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



3. (10 pts) Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$  be a linear transformation for which

$$T(\vec{e}_1) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} -4 \\ 2 \\ 1 \end{bmatrix}.$$

Write down an expression for the matrix of the linear transformation from  $\mathbb{R}^2$  to  $\mathbb{R}^3$  described as follows: first rotate all vectors in the plane by  $30^\circ$  counterclockwise, then project onto the line spanned by  $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$ , then make the resulting vector 3 times longer, and finally apply  $T$ . (You do not need to actually multiply the matrices.)

$$\vec{u} \text{ of } \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -4/5 \end{bmatrix}$$

$$T_2(\vec{x}) = A\vec{x} = T \left( \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{9}{25} & -\frac{12}{25} \\ -\frac{12}{25} & \frac{16}{25} \end{bmatrix} \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right)$$

Find  $T$ :  $A\vec{x} = \vec{b}$      $\vec{e}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$      $\vec{e}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$      $\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} a & b & | & 2 \\ c & d & | & -1 \\ e & f & | & 3 \end{bmatrix} \rightarrow \begin{bmatrix} -2a & -2b & | & -4 \\ -4c & -4d & | & -4 \\ e & f & | & 3 \end{bmatrix}$$

$$\begin{bmatrix} a & b & | & -4 \\ c & d & | & -1 \\ e & f & | & 3 \end{bmatrix}$$

$$-2ax_1 - 2by_1 = -4 \quad ax_1 + by_1 = 2$$



4. (10 pts) Suppose  $T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y - 3z \\ 2x + 3y - 5z \end{bmatrix}$ . What is  $\ker(T)$ ?

$$\ker(T) \Rightarrow T(\vec{x}) = \vec{0} \Rightarrow A\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 2 & 3 & -5 & 0 \end{array} \right]$$

$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

$$\Rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 1 & 0 \end{array} \right]$$

$$x_1 - 4x_3 = 0$$

$$x_2 + x_3 = 0$$

$$x_2 = -x_3$$

$$x_1 = 4x_3$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

$\ker(T) =$  The line spanned  
by  $\begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$ .











