Math 33A/1
Spring 2016
05/13/16
Time Limit: 50 Minutes

Name (Print): ____________________
SID Number: ____________________

Day \ T.A.	David	Casey	Adam
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

This exam contains 9 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter your name and SID number on the top of this page, cross the box correspondiing to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Use a pen to record your answers. Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors. You may use the back of this sheet for your notes ("scratch paper"). If you need additional paper, let the proctors know.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) Let $\mathcal{B}$ be the basis of $\mathbb{R}^3$ given by the vectors

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}1\\1\\2\end{bmatrix}, \vec{v}_3 = \begin{bmatrix}-2\\-1\\4\end{bmatrix}.$$

(a) Suppose the $\mathcal{B}$-coordinate vector of $\vec{x}$ is $\left[\vec{x}\right]_{\mathcal{B}} = \begin{bmatrix}0\\1\\2\end{bmatrix}$. What is $\vec{x}$?

(b) Find the $\mathcal{B}$-coordinate vector of $\vec{y} = \begin{bmatrix}1\\2\\8\end{bmatrix}$.

(c) Find the $\mathcal{B}$-matrix of $A = \begin{bmatrix}5 & -5 & 1\\-1 & 2 & 0\\-31 & 37 & -5\end{bmatrix}$.

Solution:

(a) $\vec{x} = 0 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2 + 2 \cdot \vec{v}_3 = \begin{bmatrix}1\\1\\2\end{bmatrix} + 2 \cdot \begin{bmatrix}-2\\-1\\4\end{bmatrix} = \begin{bmatrix}-3\\-1\\10\end{bmatrix}$

(b) We should find c_1, c_2, c_3 such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{y}. \tag{1}$$

There are two ways to do that:

- Staring at the numbers for a while, one simply guesses the coefficients:

$$2 \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}1\\1\\2\end{bmatrix} + \begin{bmatrix}-2\\-1\\4\end{bmatrix} = \begin{bmatrix}1\\2\\8\end{bmatrix},$$

  thus $\left[y\right]_{\mathcal{B}} = \begin{bmatrix}2\\1\\1\end{bmatrix}$.

- Alternatively, one can solve the system of linear equations (1) as we learned at the beginning of this quarter. Writing S for the matrix with column vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, the matrix form of the system (1) is $S\left[\vec{y}\right]_{\mathcal{B}} = \vec{y}$. Notice that $S = \left[\mathcal{B}\right]$ is just the change of basis matrix from $\mathcal{B}$ to the standard basis. In particular, S is invertible and thus we know two ways of solving the system: 1. Row-reducing the corresponding augmented matrix, or 2. computing S^{-1}. Which one is better?

  At this point it's worth looking ahead in the exam, noticing that we may use the matrix S^{-1} again in part (c) of the problem whereas row-reduction of the augmented matrix would be useless in part (c). Thus, although computing S^{-1} requires slightly more effort at this point, this will pay off later on. Let's compute

S^{-1} then:

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -6 & 8 & -1 \\ 0 & 1 & 0 & 5 & -6 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}$$

and we find

$$S^{-1} = \begin{bmatrix} -6 & 8 & -1 \\ 5 & -6 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

We now obtain

$$[\vec{y}]_{\mathcal{B}} = S^{-1}\vec{y} = \begin{bmatrix} -6 & 8 & -1 \\ 5 & -6 & 1 \\ -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}.$$

(c) We discussed two recipes for computing the $\mathcal{B}$-matrix of A in class. Again, the first one is quicker but involves a little amount of guessing. The second one is completely mechanical, and works especially well if we computed S^{-1} in part (b) already:

- The column vectors of $[A]_{\mathcal{B}}$ are the $\mathcal{B}$-coordinate vectors of $A\vec{v}_1$, $A\vec{v}_2$, $A\vec{v}_3$ respectively. Thus we should find these:

$$A \cdot \vec{v}_1 = \begin{bmatrix} 5 & -5 & 1 \\ -1 & 2 & 0 \\ -31 & 37 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

which is just $\vec{v}_1$, so that $[A \cdot \vec{v}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$A \cdot \vec{v}_2 = \begin{bmatrix} 5 & -5 & 1 \\ -1 & 2 & 0 \\ -31 & 37 & -5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix},$$

which is just $-\vec{v}_3$, so that $[A \cdot \vec{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$.

$$A \cdot \vec{v}_3 = \begin{bmatrix} 5 & -5 & 1 \\ -1 & 2 & 0 \\ -31 & 37 & -5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 5 \end{bmatrix},$$

which, after some trial and error, one finds to be $\vec{v}_1 + \vec{v}_3$, so that $[A \cdot \vec{v}_3]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

Putting all these together, we obtain

$$\left[A\right]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

- The second solution is to use the formula $\left[A\right]_{\mathcal{B}} = S^{-1}AS$ with S as in part (b). Using the computation of S^{-1} in part (b) we thus get

$$\begin{aligned} \left[A\right]_{\mathcal{B}} &= S^{-1}AS \\ &= \begin{bmatrix} -6 & 8 & -1 \\ 5 & -6 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & -5 & 1 \\ -1 & 2 & 0 \\ -31 & 37 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -1 \\ 1 & 2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 8 & -1 \\ 5 & -6 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 1 & -4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \end{aligned}$$

2. (10 points) (a) Find the QR-factorization of the matrix $M = \begin{bmatrix} 3 & -6 & 7 \\ 0 & -3 & 5 \\ 4 & -8 & 1 \end{bmatrix}$.

(b) Explain why $R \cdot M^{-1}$ is orthogonal.

Solution:

(a) Apply the Gram-Schmidt algorithm to the column vectors

$$\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ -8 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix}.$$

This gives us that

$$\vec{u}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}.$$

We now subtract from the second column vector its orthogonal projection onto the span of $\vec{u}_1$:

$$\begin{bmatrix} -6 \\ -3 \\ -8 \end{bmatrix} - \left(\begin{bmatrix} -6 \\ -3 \\ -8 \end{bmatrix} \cdot \vec{u}_1\right)\vec{u}_1 = \begin{bmatrix} -6 \\ -3 \\ -8 \end{bmatrix} - (-\frac{50}{5})\vec{u}_1 = \begin{bmatrix} -6 \\ -3 \\ -8 \end{bmatrix} + 10\vec{u}_1 = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}.$$

Normalizing this vector we see that

$$\vec{u}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

We now subtract from the third column vector its orthogonal projection onto the span of $\vec{u}_1$ and $\vec{u}_2$:

$$\begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} \cdot \vec{u}_1\right)\vec{u}_1 - \left(\begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} \cdot \vec{u}_2\right)\vec{u}_2 = \begin{bmatrix} 7 \\ 5 \\ 1 \end{bmatrix} - 5\vec{u}_1 - (-5)\vec{u}_2 = \begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}.$$

Normalizing this vector we see that

$$\vec{u}_3 = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ -3 \end{bmatrix}.$$

Thus,

$$Q = \frac{1}{5}\begin{bmatrix} 3 & 0 & 4 \\ 0 & -5 & 0 \\ 4 & 0 & -3 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 5 & -10 & 5 \\ 0 & 3 & -5 \\ 0 & 0 & 5 \end{bmatrix}.$$

(b) Since Q is orthogonal, we know that $(Q^{-1})^T Q^{-1} = (Q^T)^T Q^{-1} = QQ^{-1} = I$, which means that Q^{-1} is an orthogonal matrix. But since $M = QR$ we have that $Q^{-1}M = R$ and so $Q^{-1} = RM^{-1}$. Thus, RM^{-1} is an orthogonal matrix, as desired.

3. (10 points) Consider the matrix

$$A = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -2 & -1 \\ -6 & 8 & 6 \\ 8 & -12 & -10 \end{bmatrix}.$$

(a) Find a basis for $\text{image}(A)^\perp$.

(b) Compute $\text{rank}(A)$.

(c) Find all 2×2 matrices which are both orthogonal and skew-symmetric.

Solution:

(a) We have that $\text{image}(A)^\perp = \ker(A^T)$, so it suffices to find a basis for $\ker(A^T)$. We compute

$$A^T = \begin{bmatrix} -2 & 2 & -6 & 8 \\ 2 & -2 & 8 & -12 \\ 1 & -1 & 6 & -10 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And thus a basis for $\ker(A^T) = \text{image}(A)^\perp$ is given by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(b) We see from our calculation in part (a) that $\text{rref}(A^T)$ has 2 pivots, and hence $\text{rank}(A) = \text{rank}(A^T) = 2$.

(c) Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is both orthogonal and skew-symmetric. As A is skew-symmetric, we have that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A = -A^T = \begin{bmatrix} -a & -c \\ -b & -d \end{bmatrix}$$

Thus $a = -a$ and $d = -d$, so $a = d = 0$. We also must have that $b = -c$, so A is of the form $\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$. As A is orthogonal, its column vectors form an orthonormal basis for $\mathbb{R}^2$. In particular, the column vectors need to have magnitude 1, so we must have $b = \pm 1$. Thus the only matrices that are both orthogonal and skew-symmetric are $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

4. (10 points) (a) Find the least-squares solution to the system

$$\left| \begin{array}{r} x \qquad = 1 \\ -x - \quad y = 0 \\ x + 2y = 0 \\ -x + \quad y = 2 \end{array} \right|$$

(b) Compute the error for the least-squares solution in (a).

(c) Let $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ be an orthonormal basis of $\mathbb{R}^n$. Find the least-squares solution to the system

$$A\vec{x} = \vec{u}_n, \qquad \text{where } A = \begin{bmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \ldots & \vec{u}_{n-1} \\ | & | & & | \end{bmatrix}.$$

Solution:

(a) Recall the normal equation $A^T A\vec{x} = A^T\vec{b}$, where A is the coefficient matrix of the system, and $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$. Since the columns of A are linearly independent, the least squares solution is given by $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$.

We compute

$$A^T A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix},$$

so

$$(A^T A)^{-1} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix},$$

and

$$A^T\vec{b} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix},$$

so

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b} = \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

(b) The error is given by $||A\vec{x}^* - \vec{b}||$. We compute

$$A\vec{x}^* - \vec{b} = \begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} \\ 0 \\ \frac{1}{2} \\ -1 \end{bmatrix},$$

so

$$||A\vec{x}^* - \vec{b}|| = \sqrt{(-3/2)^2 + 0^2 + (1/2)^2 + (-1)^2} = \frac{\sqrt{14}}{2}.$$

(c) Again, we have $\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$. Here, $\vec{b} = \vec{u}_n$, and A is as given in the problem. Notice that

$$A^T\vec{u}_n = \begin{bmatrix} - & \vec{u}_1 & - \\ - & \vec{u}_2 & - \\ & \vdots & \\ - & \vec{u}_{n-1} & - \end{bmatrix} \vec{u}_n = \begin{bmatrix} \vec{u}_1 \cdot \vec{u}_n \\ \vec{u}_2 \cdot \vec{u}_n \\ \vdots \\ \vec{u}_{n-1} \cdot \vec{u}_n \end{bmatrix} = \vec{0}$$

since by orthogonality, $\vec{u}_i \cdot \vec{u}_n = 0$ for $i \neq n$. Thus $\vec{x}^* = (A^TA)^{-1}A^T\vec{u}_n = (A^TA)^{-1}\vec{0} = \vec{0}$.