Math 33A-1 Midterm 1 KEY Fall 2019

Name: ______________________ uid: ______________________

Section: ____________ Signature: ______________________________

Instructions:

- Unless otherwise stated, you need to justify your answer. Please show all of your work, as partial credit will be given where appropriate, **and** there may be no credit given for problems where there is no work shown.
- You have 50 minutes to complete the exam.
- All answers should be completely simplified, unless otherwise stated.
- This is a closed book and closed notes test. You may **not** use a scientific calculator. No electronics are allowed on this exam. Make sure all cell phones are silenced, put away and out of sight. If you have a cell phone out at any point, for any reason, this will constitute a violation of test policy, and you may receive a zero on this exam.
- If asked, you must show us your **bruin card** when finished with the exam.
- You may ask for scratch paper. You may use **no** other scratch paper. Please transfer all finished work onto the proper page in the test for us to grade there. We will **not** grade the work on the scratch page.
- Notice that the space left for each question is sufficient, but possibly not necessary, to answer the question.

STUDENT: PLEASE DO NOT WRITE BELOW THIS LINE. THIS TABLE IS TO BE USED FOR GRADING.

Problem	Points	Score
1	8	
2	4	
3	7	
4	4	
5	4	
6	15	
7	8	
Total	50	

1. **(8 pts)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformations determined by

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}, \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \quad S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}, \quad S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Find the matrix associated to the linear transformation $S \circ T\colon \mathbb{R}^2 \to \mathbb{R}^2$.

T has matrix $A = \begin{pmatrix}2 & 1\\1 & 0\end{pmatrix}$

S has matrix $B = \begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}$

$S \circ T$ has matrix $B \cdot A = \begin{pmatrix}0 & 0\\-1 & 0\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\-2 & -1\end{pmatrix}$

2. **(4 pts)** Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a function with

$$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}8\\5\end{pmatrix}, \quad T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}, \quad T\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix}.$$

Can T be a linear transformation? If yes, provide an example. If not, provide an explanation.

No, because $T\begin{pmatrix}2\\1\end{pmatrix} + T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}8\\5\end{pmatrix} + \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}9\\5\end{pmatrix}$

while $T\left(\begin{pmatrix}2\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right) = T\begin{pmatrix}2\\2\end{pmatrix} = \begin{pmatrix}5\\5\end{pmatrix}$

and $\begin{pmatrix}5\\5\end{pmatrix} \neq \begin{pmatrix}9\\5\end{pmatrix}$

3. For each of the following cases, write a matrix satisfying the stated property, or state why it is impossible.

(a) **(2 pts)** A is a 5×6 matrix of rank 6.

Impossible, as $\text{rank}(A) \leq 5$, cannot exceed # of rows

(b) **(2 pts)** A is a 3×2 matrix of rank 1.

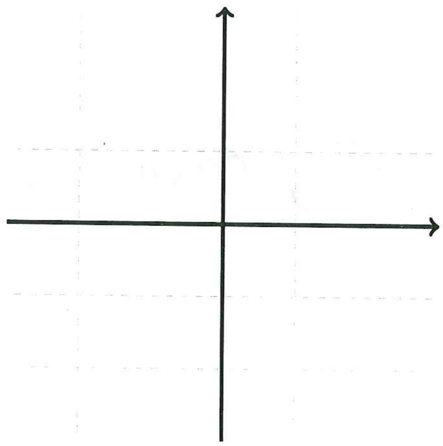
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(c) **(3 pts)** A is a 3×3 matrix with $\ker(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

We want the first and last columns of A to be opposite:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

4. **(4 pts)** In $\mathbb{R}^2$, consider the subset W consisting of the two coordinate axes (the solid lines in the picture below). Is W a linear subspace of $\mathbb{R}^2$? Motivate your answer.



No, $\vec{e_1}$ is in W

$\vec{e_2}$ is in W

But $\vec{e_1} + \vec{e_2}$ is not in W

5. **(4 pts)** Consider the following vectors in $\mathbb{R}^4$: $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 7 \\ 3 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 5 \\ 4 \end{pmatrix}$, $\vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

Is the vector $\begin{pmatrix} 45 \\ 18 \\ 3 \\ 1 \end{pmatrix}$ in $\text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$? Motivate your answer.

If $\vec{v}$ is in $\text{span}(\vec{v_1}, \vec{v_2}, \vec{v_3})$

$$\vec{v} = c_1\vec{v_1} + \vec{c_2}\vec{v_2} + \vec{c_3}\vec{v_3} = \begin{pmatrix} c_1 + 3c_2 \\ c_1 + 3c_2 \\ 7c_1 + 5c_2 + c_3 \\ 3c_1 + 4c_2 \end{pmatrix}$$

So, the first two entries of $\vec{v}$ are both c_1+3c_2, they are equal

But $45 \neq 18$. So, $\vec{v}$ is not in the span.

7. Let $T\colon \mathbb{R}^5 \to \mathbb{R}^4$ be the linear transformation with associated matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 6 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) **(3 pts)** Write the image of T as the span of a set of vectors.

$$\mathrm{im}(A) = \mathrm{span}\left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\6\\1\\0\end{pmatrix}, \begin{pmatrix}3\\4\\5\\0\end{pmatrix} \right)$$

(b) **(5 pts)** Find a basis for the image of T. Motivate why the set you exhibit is a basis (you may refer to a theorem, or argue directly).

$$\mathrm{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 4-6\cdot 5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -26 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1st, 3rd and 4th columns have pivots

So basis is $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\6\\1\\0\end{pmatrix}$

6. Let A be the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}.$$

(a) **(10 pts)** Using row operations, find (if possible) the inverse matrix of A.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right) \longrightarrow \begin{array}{c} \mathrm{I} \\ \mathrm{II}-\mathrm{I} \\ \mathrm{III}-\mathrm{I} \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right) \longrightarrow \begin{array}{c} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III}-2\mathrm{II} \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\begin{array}{c} \mathrm{I}-\mathrm{III} \\ \mathrm{II}-2\mathrm{III} \\ \mathrm{III} \end{array} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \qquad \begin{array}{c} \mathrm{I}-\mathrm{II} \\ \mathrm{II} \\ \mathrm{III} \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

(b) **(5 pts)** Find all the solutions of the following linear system:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}.$$

$$\vec{x} = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ -3 & 5 & 2 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3+2 \\ -3+4 \\ 1+2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$$

unique solution