Midterm 2
Linear Algebra and Applications
(Math 33A-001)

Answer the questions in the spaces provided. If you run out of room for an answer, please continue on the back of the page. **Show all of your work.**

Name: [redacted]

Question:	1	2	3	Total
Points:	5	5	10	20
Score:	5	3	10	18

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$\mathbb{R}^2 \to \mathbb{R}^3$

1. $\boxed{\text{5 points}}$ Let A be a $\underbrace{3 \times 2}$ matrix with column vectors $\vec{v}_1, \vec{v}_2$, i.e.,

$$A = \begin{pmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{pmatrix}$$

$C_{n \times n} = \begin{bmatrix} | & | & | \\ \vec{v} & \vec{v}_1 & \vec{v}_2 \\ | & | & | \end{bmatrix}$

lin. indep.

Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^3$. You are told that $\vec{v}, \vec{v}_1, \vec{v}_2$ form a basis of $\mathbb{R}^3$. Then what is the rank of the matrix B with column vectors $\vec{v}$, $2\vec{v}+\vec{v}_1$, $2\vec{v}+3\vec{v}_2$,

invertible

i.e., $B = \begin{pmatrix} | & | & | \\ \vec{v} & 2\vec{v}+\vec{v}_1 & 2\vec{v}+3\vec{v}_2 \\ | & | & | \end{pmatrix}$?

$\dim(\operatorname{im} B) + \dim(\ker B) = 3$

$\operatorname{rank} B + \dim(\ker B) = 3$

(Remark: An answer without proper justification ears you a **'0' point**. You must justify your answer.)

$$c_1\vec{v} + c_2(2\vec{v}+\vec{v}_1) + c_3(2\vec{v}+3\vec{v}_2) = 0$$

$$\vec{v}\underbrace{(c_1+2c_2+2c_3)}_{A} + \vec{v}_1\underbrace{(c_2)}_{B} + \vec{v}_2\underbrace{(3c_3)}_{C} = 0$$

$\therefore \quad A\vec{v} + B\vec{v}_1 + C\vec{v}_2 = 0$

Since $\vec{v}, \vec{v}_1, \vec{v}_2$ form a basis of $\mathbb{R}^3$, by definition of a basis, they must be linearly independent. Thus, we can only have the trivial solution $A = B = C = 0$.

$c_1 + 2c_2 + 2c_3 = 0$: Plugging in $c_2 = c_3 = 0$: $\boxed{c_1 = 0}$

$\boxed{c_2 = 0}$

$3c_3 = 0 \therefore \boxed{c_3 = 0}$

5

Since we get the trivial solution $c_1 = c_2 = c_3 = 0$, $\vec{v}$, $2\vec{v}+\vec{v}_1$, and $2\vec{v}+3\vec{v}_2$ must be linearly independent. Therefore, the $\dim(\ker B) = 0$, since there are no dependent (OR redundant) vectors. By the rank-nullity theorem, $\operatorname{rank} B + \dim(\ker B) =$ # columns of B. Therefore since $\dim(\ker B) = 0$,

$\boxed{\operatorname{rank} B = 3}$.

3

\# vectors in basis of V

lin. indep, span V

2. $\boxed{\text{5 points}}$ Let V be a subspace of $\mathbb{R}^n$ of $\dim V = m$ and $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_m\}$ a basis of V. Show that a vector $\vec{\mathbf{x}} \in \mathbb{R}^n$ is orthogonal to V if it is orthogonal to all the vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_m\}$.

By definition of a basis, $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are all linearly independent. Vectors are orthogonal to each other if their dot product equals zero.

By definition of a basis, we can also say that $\vec{v}_1, \ldots, \vec{v}_m$ span V.

Since $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent,

$$c_1\vec{v}_1 + \ldots + c_m\vec{v}_m = 0$$

such that we get the trivial relation $c_1 = \ldots = c_m = 0$.

If we take the dot product of $\vec{x}$ and the span of V, we get

$$c_1\vec{x}\cdot\vec{v}_1 + \ldots + c_m\vec{x}\cdot\vec{v}_m = \vec{x}\cdot(c_1\vec{v}_1 + \ldots + c_m\vec{v}_m) = 0$$

Thus, $\vec{x} \perp v_i$ for $i = 1, \ldots, m$.

3. $\boxed{\text{10 points}}$ For each of the following statements, determine whether it is true or false.

2 1. Let A and B be two $n \times n$ matrices. You are told that $A + B$ is **invertible**. Then A and B are necessarily invertible.

2 2. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of **linearly independent** vectors of $\mathbb{R}^n$. Let $T = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_l\}$ be another set of **linearly independent** vectors of $\mathbb{R}^n$. Then $S \cup T$ is always a linearly independent set of vectors.

→ lin. indep., span $\mathbb{R}^n$

2 3. Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be a set of **orthonormal** vectors of $\mathbb{R}^n$. Let $T = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_l\}$ be another set of **orthonormal** vectors of $\mathbb{R}^n$. Then $S \cup T$ is always an orthonormal set of vectors.

2 4. Let A and B be two $n \times n$ matrices such that $\text{rank}(AB) < n$. You are told that A is **invertible**. Then B is never invertible.

2 5. Let A and B be two $n \times n$ matrices such that $\text{rank}(A) = \text{rank}(B)$. Then $\text{Ker}(A) = \text{Ker}(B)$ always holds.

(**Remark:** Only a 'True' or 'False' answer without any justification ears you a **'0' point**. You must justify your answer.)

1) $\boxed{\text{False}}$: Prove by ~~contradiction~~ counter example

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \quad \det(A+B) = -2 \neq 0 \quad \therefore \text{invertible}$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \det A = 0 \quad \therefore \text{noninvertible}$$

2) $\boxed{\text{False}}$ If $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and $T = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$\vec{v}_1$, $\vec{v}_2$ lin. indep. ; $\vec{w}_1$, $\vec{w}_2$ lin. indep

$$\vec{w}_2 = \vec{v}_1 + \vec{v}_2$$

$\therefore$ $S \cup T$ is not alway a linearly indep. set of vectors.

3) $\boxed{\text{False}}$ Orthonormal vectors are linearly independent and span $\mathbb{R}^n$. Since no unit vector can be a linear combination of other unit vectors and orthonormal vectors are perpendicular to each other, the union of S and T does not ensure that the vectors in the respective sets are still perpendicular, e.g. $S = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ and T contain $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

this vector is not perpendicular to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow S \cup T$ ∴ not orthonormal

4) $\boxed{\text{True}}$ An invertible matrix $A_{n \times n}$ must have $\text{rank}(A) = n$. When you multiply two invertible matrices, you get another invertible matrix. Since $A_{n \times n}$ & $B_{n \times n}$, $(AB)_{n \times n}$. If A and B are invertible than $(AB)_{n \times n}$ must be invertible and $\text{rank}(AB)$ must be n. Since A is invertible, the only way that AB has a rank less than n is if B is not invertible and thus AB is not invertible.

5) $\boxed{\text{False}}$ By the rank-nullity theorem,

$$\text{rank}(A) + \dim(\ker A) = n$$
$$\text{rank}(B) + \dim(\ker B) = n$$

$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ rref $= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$v_1 = -2v_2$
$v_2 = t \Rightarrow t\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ rref $= \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$

$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$w_1 = -3w_2 \Rightarrow$ span of $s \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

Since $\text{rank}(A) = \text{rank}(B)$, $\dim(\ker A) = \dim(\ker B)$. However, the number of vectors that form the basis of $\ker(A)$ & $\ker(B)$ being the same does NOT entail that the vector of their respective bases must be equal. For example, $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ ∴ rref$A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ ∴ $\text{rank}(A) = 1$ and $\ker(A) =$ span of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. $B = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}$ ∴ rref$B = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$ ∴ $\text{rank}(B) = 1$ and $\ker(B) =$ span of $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$. ∴ $\ker A \neq \ker B$.

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