(1)

Solutions for Midterm 1

1) We use the Gauss-Jordan algorithm.
The augmented coefficient matrix A of the system is:

$$\left(\begin{array}{cccc|c} -2 & 4 & 0 & 2 & -6 \\ 3 & -6 & 3 & 3 & 3 \\ 3 & -6 & 1 & -1 & 7 \end{array}\right) \sim \begin{array}{l} \left(-\frac{1}{2}\right)\cdot \mathrm{I} \\ \frac{1}{3}\cdot \mathrm{II} \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 3 \\ 1 & -2 & 1 & 1 & 1 \\ 3 & -6 & 1 & -1 & 7 \end{array}\right) \sim \begin{array}{l} \mathrm{II} + (-1)\,\mathrm{I} \\ \mathrm{III} + (-3)\,\mathrm{I} \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{array}\right) \sim \begin{array}{l} \\ \mathrm{III} + (-1)\,\mathrm{II} \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \mathrm{rref}(A)$$

So x_1, x_3 are the pivot variables, and x_2, x_4 are the free variables.
We set $x_2 = s$, $x_4 = t$, where $s, t \in \mathbb{R}$ are arbitrary. Then

$$x_1 = 3 + 2x_2 + x_4 = 3 + 2s + t,$$
$$x_3 = -2 - 2x_4 = -2 - 2t.$$

(2) We conclude that the general solution of system is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3+2s+t \\ s \\ -2-2t \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix},$$

where $s, t \in \mathbb{R}$ are arbitrary.

2) We use the Gauss-Jordan algorithm with the 3×3-identity matrix as augentation:

$$\left(\begin{array}{ccc|ccc} -2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \sim \quad \underline{\mathrm{I}} \longleftrightarrow \underline{\mathrm{II}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 1 & 0 & 1 & 0 & 0 \\ 4 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \sim \begin{array}{l} \underline{\mathrm{II}} + 2 \cdot \underline{\mathrm{I}} \\ \underline{\mathrm{III}} + (-4) \cdot \underline{\mathrm{I}} \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & -2 & -3 & 0 & -4 & 1 \end{array}\right) \sim \quad \underline{\mathrm{III}} + 2 \cdot \underline{\mathrm{II}}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right) \sim \begin{array}{l} \underline{\mathrm{I}} + (-1) \underline{\mathrm{III}} \\ \underline{\mathrm{II}} + (-2) \underline{\mathrm{III}} \end{array}$$

(3) $$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{rref(A)} \left| \begin{matrix} -2 & 1 & -1 \\ -3 & 2 & -2 \\ 2 & 0 & 1 \end{matrix} \right)$$

Since the reduced row-echelon form of A is the 3×3-identity matrix, we conclude that A is invertible with inverse

$$A^{-1} = \begin{pmatrix} -2 & 1 & -1 \\ -3 & 2 & -2 \\ 2 & 0 & 1 \end{pmatrix}.$$

3) a) The inverse of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad-bc \neq 0$ is given by $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

In this case,

$A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and $ad-bc = 2\cdot2 - 3\cdot1 = 1$.

So $A^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

b) If we multiply the equation $C = ABA$ from the left and from the right by A^{-1}, then we obtain

$$A^{-1}CA^{-1} = A^{-1}ABAA^{-1} = I_2BI_2 = B.$$

So $B = A^{-1}CA^{-1}$!

(4) c) By part (b) we have

$$B = A^{-1} C A^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -10 & 16 \\ 7 & -11 \end{pmatrix}.$$