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Solutions for Midterm 2

1) a) We first use the Gauss-Jordan algorithm to find a basis of the solution space V of the homogeneous linear system.

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & -4 & -9 \end{pmatrix} \overset{II-I}{\sim} \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 2 & -4 & -8 \end{pmatrix} \overset{\frac{1}{2}\cdot II}{\sim}$$

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & -2 & -4 \end{pmatrix} \overset{I+II}{\sim} \begin{pmatrix} 1 & 0 & -2 & -5 \\ 0 & 1 & -2 & -4 \end{pmatrix}$$

The free variables are x_3, x_4. We set $x_3 = s$, $x_4 = t$. Then

$x_1 = 2x_3 + 5x_4 = 2s + 5t$,

$x_2 = 2x_3 + 4x_4 = 2s + 4t$.

So the general solution of the homogeneous system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2s + 5t \\ 2s + 4t \\ s \\ t \end{pmatrix} = s\begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 5 \\ 4 \\ 0 \\ 1 \end{pmatrix}, \quad s,t \in \mathbb{R} \text{ arb.}$$

A basis of V is given by

$$v_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 5 \\ 4 \\ 0 \\ 1 \end{pmatrix}.$$

To find an orthonormal basis, we use the Gram-Schmidt process.

② $\|v_1\| = \sqrt{2^2+2^2+1^2} = \sqrt{9} = 3$.

So
$$u_1 = \frac{1}{3} v_1 = \frac{1}{3}\begin{pmatrix}2\\2\\1\\0\end{pmatrix}.$$

$u_2' = v_2 - (v_2 \cdot u_1)\, u_1$.

$$v_2 \cdot u_1 = \begin{pmatrix}5\\4\\0\\1\end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix}2\\2\\1\\0\end{pmatrix} = \frac{1}{3}(10+8) = 6.$$

So
$$u_2' = \begin{pmatrix}5\\4\\0\\1\end{pmatrix} - 6 \cdot \frac{1}{3}\begin{pmatrix}2\\2\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-2\\1\end{pmatrix}$$

$\|u_2'\| = \sqrt{1^2+(-2)^2+1^2} = \sqrt{6}$.

$$u_2 = \frac{1}{\|u_2'\|} \cdot u_2' = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}.$$

Conclusion: An orthonormal basis of V is given by
$$u_1 = \frac{1}{3}\begin{pmatrix}2\\2\\1\\0\end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\0\\-2\\1\end{pmatrix}.$$

b) Since V has a basis consisting of two elements, its dimension is equal to 2, i.e., $\dim V = 2$.

③ 2) a) The vectors $v_1, \dots, v_k \in \mathbb{R}^n$ are called linearly independent if the equation

$$a_1 v_1 + \cdots + a_k v_k = 0$$

has the only solution $a_1 = a_2 = \cdots = a_k = 0$.

b) The vectors $v_1, \dots, v_k$ in a subspace V of $\mathbb{R}^n$ span V if every vector $v' \in V$ can be written as a linear combination of the vectors $v_1, \dots, v_k$.

c) The vectors $v_1, \dots, v_k$ form a basis of a subspace V of $\mathbb{R}^n$ if they are linearly independent and span V.

d) The vectors v_1, v_2, v_3 are <u>not</u> linearly independent, because we have the non-trivial representation of the zero-vector given by

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 1 \cdot v_1 + 1 \cdot v_2 - v_3.$$

They vectors v_1, v_2, v_3 span $\mathbb{R}^2$, because every vector in $\mathbb{R}^2$ can be written as a linear combination of v_1, v_2, v_3; indeed, if $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ is arb., then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left(\frac{x+y}{2}\right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(\frac{x-y}{2}\right) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \left(\frac{x+y}{2}\right) v_1 + \left(\frac{x-y}{2}\right) v_2 + 0 \cdot v_3.$$

The vectors v_1, v_2, v_3 do <u>not</u> form a basis of $\mathbb{R}^2$, because they are not linearly independent.

④ 3) a) Note: $\sin(0) = 0$, $\sin(\pi/2) = 1$, $\sin(3\pi/2) = -1$.

We would like to fit the data vector

$$y = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \text{ by } a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$ corresponding to

$$y = a + b \sin(x\pi/2)$$

To get the optimal fit according to the method of least squares, we let

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$ and solve the normal equation

$$A^T A x = A^T y, \text{ where } x = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2,$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

$$A^T y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

So the normal equation is

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ equiv. } \begin{matrix} 3a = 3, \\ 2b = 4. \end{matrix}$$

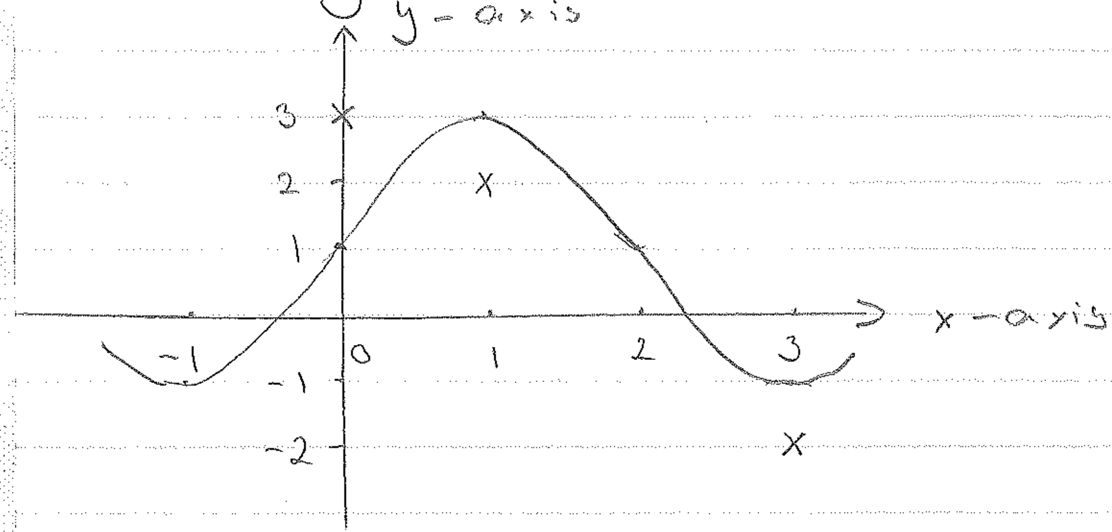
Hence $\underline{a = 1, \ b = 2}$.

Conclusion: The function

$$y = 1 + 2\sin(x\pi/2)$$

gives the optimal fit for the given data points.

⑤ b) Sketch of data points and graph of the function $y = 1+2\sin(x\pi/2)$:



4) a) If a_{ij} is the entry of the matrix A on the i-th row and j-th column, then
$a_{ij} = a_{ji}$, because A is symmetric.
Since A is also antisymmetric, we also have
$a_{ij} = -a_{ji}$.
It follows that
$a_{ij} = -a_{ji} = -a_{ij}$, and so $2a_{ij} = 0$.
This implies $a_{ij} = 0$.
<u>Conclusion</u>: All the entries of the matrix A are equal to 0.

b) We have to prove existence and uniqueness of the representation.

<u>1. Existence</u>: Let M be an arbitrary $n \times n$-matrix.

Define $S = \frac{1}{2}(M+M^T)$, $A = \frac{1}{2}(M-M^T)$.

(b) Then S is symmetric, because
$$S^T = \frac{1}{2}(M+M^T)^T = \frac{1}{2}(M^T+M) = S,$$
and A is antisymmetric, because
$$A^T = \frac{1}{2}(M-M^T)^T = \frac{1}{2}(M^T-M) = -A.$$
Here we used that
$$(M^T)^T = M.$$
Since
$$S+A = \frac{1}{2}(M+M^T) + \frac{1}{2}(M-M^T) = M,$$
we obtain a representation of M as desired. □

2. Uniqueness: Suppose M is an $n\times n$-matrix and $M = S_1 + A_1$ and $M = S_2 + A_2$, where S_1, S_2 are symmetric and A_1, A_2 are antisymmetric.
Then $S_1 + A_1 = S_2 + A_2$ and so
$$S_1 - S_2 = A_2 - A_1.$$
Let $N = S_1 - S_2 = A_2 - A_1$. Then N is both symmetric and antisymmetric.
Indeed,
$$N^T = (S_1 - S_2)^T = S_1^T - S_2^T = S_1 - S_2 = N,$$
and
$$N^T = (A_2 - A_1)^T = A_2^T - A_1^T = -A_2 + A_1 = -N.$$
By part (a) we conclude that $N = 0$.

(7) So $N = S_1 - S_2 = 0$, and $N = A_2 - A_1 = 0$.
This implies $S_1 = S_2$ and $A_1 = A_2$.
The uniqueness of the representation of M with the stated properties follows. $\square$

5) a) The kernel $\mathrm{Ker}(A)$ of an $n \times k$-matrix A is the set of all vectors $x \in \mathbb{R}^k$ such that $Ax = 0$.

b) Suppose $B = (v_1 \cdots v_k)$ is a matrix with columns $v_1, \dots, v_k$. Then for $x_1, \dots, x_k \in \mathbb{R}$ we have

$$x_1 v_1 + \cdots + x_k v_k = Bx, \text{ where } x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \in \mathbb{R}^k.$$

This implies that the column space of B consists precisely of all vectors y of the form $y = Bx$ with $x \in \mathbb{R}^k$.
Now let A be an arbitrary matrix, say a $n \times k$-matrix. Then A^T is a $k \times n$-matrix.

Let x be an arbitrary vector in $\mathrm{Ker}(A)$ and y be an arbitrary vector in the column space of A^T. By the initial remark, y can be written in the form $y = A^T u$, where $u \in \mathbb{R}^n$.

Hence $$(y \cdot x) = y^T x = (A^T u)^T x = (u^T A)x = u^T(Ax) = 0.$$

Here we used $(A^T)^T = A$ and $Ax = 0$, because $x \in \mathrm{Ker}(A)$. It follows that y and x are orthogonal, as desired. $\square$