Math 33A, Lecture 2
Spring 2016
05/16/16
Time Limit: 50 Minutes

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Tuesday	~~2A~~	2C	2E
Thursday	2B	2D	2F

This exam contains 8 pages (including this cover page) and 4 problems. Check to see if any pages are missing. Enter your name, SID number, and signature on the top of this page, cross the box corresponding to your discussion section, and put your initials on the top of every page, in case the pages become separated. Also, have your photo ID on the desk in front of you during the exam.

Use a pen to record your final answers. Calculators or computers of any kind are not allowed. You are not allowed to consult any other materials of any kind, including books, notes and your neighbors.
You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.** A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the last page; clearly indicate when you have done this. If you need additional paper, let the proctors know.

Of course, if you have a question about a particular problem, please raise your hand and one of the proctors will come and talk to you.

Problem	Points	Score
1	20	20
2	20	20
3	20	20
4	20	18
Total:	80	~~78~~ 80

1. (20 points)

Let $A = \begin{pmatrix} 1 & -1 & k-1 \\ 2 & 0 & k+1 \\ k & 1 & 2 \end{pmatrix}$

(a) Compute $det(A)$.

(b) Find all value(s) of k for which A is invertible.

(c) For $k = 1$, find a basis of $im(A)^{\perp}$.

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a) $\det(A) = \cancel{-2k^2 + k + 1}$ $\boxed{-k^2+1}$

b) Invert. when $\det(A) \neq 0$

$\cancel{-2k^2+k+1 \neq 0}$ $\quad -k^2+1 \neq 0$

$\cancel{-(2k^2-k-1) \neq 0}$ $\quad -(k^2-1) \neq 0$

$k^2 \neq 1$

$\cancel{k^2 - \frac{1}{2}k - \frac{1}{2} \neq 0}$ $\quad k \neq 1, -1$

$\cancel{(k-1)(k+\frac{1}{2}) \neq 0}$

$\cancel{k \neq 1}$

$\cancel{k \neq \frac{-1}{2}}$

A is invertible for all other k

$\boxed{\text{A is invertible for all } k \text{ except } k=1, \cancel{k=-1/2}\ k=-1}$

$\det(A) = $ $\begin{pmatrix} 1 & -1 & k-1 \\ 2 & 0 & k+1 \\ k & 1 & 2 \end{pmatrix} \begin{matrix} 1 & -1 \\ 2 & 0 \\ k & 1 \end{matrix}$

$0 - k^2 - k + 2k - 2$

$-0 - k - 1 + 4$

$-k^2 + 1 \neq 0$

$+k^2 \neq +1$

$k \neq \pm 1$

$-k^2 - k - k + 2k - 2 - 1 + 4$

$-k^2 - 2 - 1 + 4$

$-k^2 + 1$

$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix} \begin{matrix} 1 & -1 \\ 2 & 0 \\ 1 & 1 \end{matrix}$

$-2 - 2 + 4$

0

$-(1) + 1 = 0$

$\cancel{\det(A) = 0 + (-1)(k+1)(k) + (k-1)(2) - k(k-1) - (k+1) - (-4)}$

$\cancel{-k^2 - k + 2k - 2 - k^2 + k - k - 1 + 4}$

a) $\boxed{-2k^2 + k + 1}$

c) $im(A)^{\perp} = \ker(A^T)$ $\quad A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 2 \end{pmatrix}$

$\ker(A^T)$: $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 0 & 2 & 2 \end{pmatrix} \xrightarrow{+\mathrm{I}} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix} \xrightarrow[-\mathrm{II}]{\div 2} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{-2\mathrm{II}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

let $x_3 = t$

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ -x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

basis for $im(A)^{\perp}$

$\Rightarrow \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$0 + (-1)(k+1)(k) + 2(k-1) \quad = -k^2 + 1 -$

$-0 - (k+1) + 4$

$-k^2 - k - 2$

$+2k$

$-k - 1$

$+4$

20

2. (20 points) Fit a linear data function of the form $f(t) = c_0 + c_1 t$ to the data points

$$(0,3), (1,2), (2,6), (-1,2).$$

$f(t) = c_0 + c_1 t$

$f(0) = c_0 = 3$

$f(1) = c_0 + c_1 = 2$

$f(2) = c_0 + 2c_1 = 6$

$f(-1) = c_0 - c_1 = 2$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 3 \\ 2 \\ 6 \\ 2 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix} \qquad A^T\vec{b} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \end{pmatrix}$$

$$\begin{array}{r} 39 \\ -12 \\ \hline 27 \end{array}$$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}$$

$1+4+1=6$ ✓

$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \end{pmatrix}$$

$$\begin{array}{r} 13 \\ -24 \end{array} \qquad \begin{array}{r} 24 \\ -13 \\ \hline 11 \end{array}$$

$4x_1 + 2x_2 = 13$

$2x_1 + 6x_2 = 12$ $\rightarrow$

$-10x_2 = -11 \qquad x_2 = 11/10$

$2x_1 + 6(11/10) = 12$

$2x_1 + 66/10 = 12$

$2x = \frac{120-66}{10} = \frac{54}{10}$

$x_1 = \frac{54}{20} = \frac{27}{10}$

$12 - 39$

$-10x_1 = -27$

$x_1 = \frac{27}{10}$

$$\boxed{f(t) = \frac{27}{10} + 11/10\, t}$$

$$\vec{x}^* = \begin{pmatrix} 27/10 \\ 11/10 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

$\frac{27}{10} + \frac{22}{10} = \frac{49}{10} = 5$

$\frac{16}{10} = 1.6$

$\frac{38}{}$ 4

3. (20 points)

(a) Find the β-matrix $B = [T]_\beta$ of T, where $\beta = \left(\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right)$ and

$$T\left(\begin{pmatrix}x\\y\end{pmatrix}\right) = \begin{pmatrix}3x-y\\5x-2y\end{pmatrix} \qquad A = \begin{pmatrix}3 & -1\\5 & -2\end{pmatrix}$$

(b) Let

$$\gamma = \left(\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}, \begin{pmatrix}-3/\sqrt{20}\\3/\sqrt{20}\\-1/\sqrt{20}\\1/\sqrt{20}\end{pmatrix}, \begin{pmatrix}1/2\\1/2\\-1/2\\-1/2\end{pmatrix}, \begin{pmatrix}1/\sqrt{20}\\-1/\sqrt{20}\\-3/\sqrt{20}\\3/\sqrt{20}\end{pmatrix}\right).$$

We are told that γ is an orthonormal basis of $\mathbb{R}^4$. Find the γ-coordinate vector

$$\left[\begin{pmatrix}1\\1\\2\\2\end{pmatrix}\right]_\gamma$$

a)

$S = \begin{pmatrix}1 & 1\\2 & 1\end{pmatrix}$ $A = \begin{pmatrix}3 & -1\\5 & -2\end{pmatrix}$

$S^{-1} = \frac{1}{-1}\begin{pmatrix}1 & -1\\-2 & 1\end{pmatrix} = \begin{pmatrix}-1 & 1\\2 & -1\end{pmatrix}$

$B = S^{-1}AS$

$B = \begin{pmatrix}-1 & 1\\2 & -1\end{pmatrix}\begin{pmatrix}3 & -1\\5 & -2\end{pmatrix} S$

$\begin{pmatrix}2 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\2 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}$

$B = \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}$ ✓

A

$\begin{pmatrix}3 & -1\\5 & -2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

$= 1v_1 = \begin{pmatrix}0\\1\end{pmatrix}$ ✓

$\begin{pmatrix}3 & -1\\5 & -2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}2\\3\end{pmatrix}$

$= \begin{pmatrix}1\\1\end{pmatrix}$ ✓

b)

$\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}\begin{pmatrix}1\\1\\2\\2\end{pmatrix} = 1+1+1 = \underline{3}\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}3/2\\3/2\\3/2\\3/2\end{pmatrix}$

$\begin{pmatrix}-3/\sqrt{20}\\3/\sqrt{20}\\-1/\sqrt{20}\\1/\sqrt{20}\end{pmatrix} = \frac{-3}{\sqrt{20}} + \frac{3}{\sqrt{20}} - \frac{2}{\sqrt{20}} + \frac{2}{\sqrt{20}} = 0$

$\begin{pmatrix}1/2\\1/2\\-1/2\\-1/2\end{pmatrix}\begin{pmatrix}1\\1\\2\\2\end{pmatrix} = \frac{1}{2} + \frac{1}{2} - 1 - 1$

$1-1-1 = -1$

$\begin{pmatrix}1/\sqrt{20}\\-1/\sqrt{20}\\-3/\sqrt{20}\\3/\sqrt{20}\end{pmatrix}\begin{pmatrix}1\\1\\2\\2\end{pmatrix} = \frac{1}{\sqrt{20}} - \frac{1}{\sqrt{20}} - \frac{6}{\sqrt{20}} + \frac{6}{\sqrt{20}} = 0$

$\left[\begin{pmatrix}1\\1\\2\\2\end{pmatrix}\right]_\gamma = \begin{pmatrix}3\\0\\-1\\0\end{pmatrix}$

✓ $= \begin{pmatrix}3\\0\\-1\\0\end{pmatrix}$

4. (20 points)

(a) For each of the following statements, circle T for True, F for False.

T F Every subspace $V \subset \mathbb{R}^n$ has an orthonormal basis.

T F The matrix $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is orthogonal. $\|v_1\| = \sqrt{1+1} = \sqrt{2}$

x2 T F The exists a matrix A so that $ker(A) = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ ker(A) is a subspace $\vec{0} \in ker(A)$

3×2 2×2 3×2

T F The size of the matrix R in a QR factorization of $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 2 \end{pmatrix}$ is 2×2.

$Q = m\times p$ $\therefore Q$ is $3\times p$ not $2\times p$
$R = n\times q$
$A = m\times q$
$A = QR$
$m = 3$
$q = 2$
R is $2\times q$
r_{ij} = sq.
2×2 ✓

T F The vectors $\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ form a basis of $\mathbb{R}^4$

At most 4 form a basis. They span $\mathbb{R}^4$ at most.

(b) Let $\alpha = (\vec{x_1}, \vec{x_2})$ and $\beta = (\vec{y_1}, \vec{y_2})$ be two bases of $\mathbb{R}^2$. Suppose that $[\vec{x_1}]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $[\vec{x_2}]_\beta = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

Find the β-coordinate vector $[\vec{x_1} + 2\vec{x_2}]_\beta$, and the α-coordinate vectors $[\vec{y_1}]_\alpha$ and $[\vec{y_2}]_\alpha$.

$\alpha = (\vec{x_1}, \vec{x_2})$ $S_A = (x_1 \; x_2)$
$\beta = (y_1, y_2)$ $S_B = (y_1 \; y_2)$

$[x_1]_\beta = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = y_1 + 2y_2$
$[x_2]_\beta = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = y_1 + 3y_2$

$S_A = \begin{pmatrix} x_{1a} & x_{2a} \\ x_{1b} & x_{2b} \end{pmatrix}$
$S_A^{-1} = \frac{1}{x_{1a}x_{2b} - x_{2a}x_{1b}} \begin{pmatrix} x_{2b} & -x_{2a} \\ -x_{1b} & x_{1a} \end{pmatrix}$

Find $[\vec{x_1} + 2x_2]_\beta = [x_1]_\beta + 2[x_2]_\beta = (y_1 + 2y_2) + 2(y_1 + 3y_2) = y_1 + 2y_1 + 2y_2 + 6y_2$

$= 3y_1 + 8y_2$

$= \begin{pmatrix} 3 \\ 8 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 3 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$ ✓

find $[y_1]_\alpha$, $[y_2]_\alpha$

$[y_1]_\alpha = c_1 x_1 + c_2 x_2$
$[y_1]_\alpha = c_1(y_1 + 2y_2) + c_2(y_1 + 3y_2)$
$c_1 y_1 + 2c_1 y_2 + c_2 y_1 + 3c_2 y_2$
$(c_1 + c_2) y_1 + (2c_1 + 3c_2) y_2$
$2c_1 + 3c_2 = 0$
$c_1 + c_2 = 1$
$c_2 = -2$
$c_1 = 3$

$[y_2]_\alpha = c_3 x_1 + c_4 x_2$
$(c_3 + c_4) y_1 + (2c_3 + 3c_4) y_2$
$c_3 + c_4 = 0$
$2c_3 + 3c_4 = 1$
$c_4 = 1$
$c_3 = -1$

$[y_2]_\alpha = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$[y_1]_\alpha = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

$[\vec{x_1} + 2\vec{x_2}]_\beta = \begin{pmatrix} 3 \\ 8 \end{pmatrix}$

$[\vec{y_1}]_\alpha = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $[\vec{y_2}]_\alpha = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$