MATH 33A - MIDTERM 1

Discussion session: 2E

Question 1: 17 / 25

Question 2: 25 / 25

Question 3: 20 / 25

Question 4: 18 / 25

Total: 80 / 100

Problem 1. *(25 points)*

(a) *(10 points)* Let A be a square and invertible matrix. Show that

$$\boxed{(A^T)^{-1} = (A^{-1})^T.}$$

(b) *(15 points)* Find the inverse of the matrix

$$B = \begin{bmatrix}0 & 1 & 2\\ 1 & 0 & 3\\ 4 & -3 & 8\end{bmatrix}.$$

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10 (a)

(a) $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$

$A^T=\begin{bmatrix}a & c\\ b & d\end{bmatrix}$

$(A^T)^T = A$

$(A^T)^{-1} = \begin{bmatrix}d & -c\\ -b & a\end{bmatrix}\frac{1}{ad-bc}$

$(A^{-1}) = \begin{bmatrix}d & -b\\ -c & a\end{bmatrix}\frac{1}{ad-bc}$

$(A^{-1})^T = \begin{bmatrix}d & -c\\ -b & a\end{bmatrix}\frac{1}{ad-bc}$

Need to show this!

Both of these are equal. This 2x2 exercise can be generalized an nxn square matrix. So, $(A^T)^{-1}=(A^{-1})^T$.

(b) $B=\begin{bmatrix}0 & 1 & 2\\ 1 & 0 & 3\\ 4 & -3 & 8\end{bmatrix}$

$$\left[\begin{array}{ccc|ccc}0 & 1 & 2 & 1 & 0 & 0\\ 1 & 0 & 3 & 0 & 1 & 0\\ 4 & -3 & 8 & 0 & 0 & 1\end{array}\right] \xrightarrow{①} \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 4 & -3 & 8 & 0 & 0 & 1\end{array}\right]$$

$$\xrightarrow{②} \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 0 & -3 & -4 & 0 & -4 & 1\end{array}\right] \xrightarrow{③} \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 2 & 1 & 0 & 0\\ 0 & 0 & 2 & 3 & -4 & 1\end{array}\right]$$

$$\xrightarrow{④} \left[\begin{array}{ccc|ccc}1 & 0 & 3 & 0 & 1 & 0\\ 0 & 1 & 0 & -2 & 4 & -1\\ 0 & 0 & 2 & 3 & -4 & 1\end{array}\right] \xrightarrow{⑤} \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2}\\ 0 & 1 & 0 & -2 & 4 & -1\\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2}\end{array}\right]$$

So, $B^{-1}=\begin{bmatrix}-\frac{9}{2} & 7 & -\frac{3}{2}\\ -2 & 4 & -1\\ \frac{3}{2} & -2 & \frac{1}{2}\end{bmatrix}$

Answer.

Steps Taken

1) Switch rows 1 and 2.
2) Multiply 1st row by -4 and add it to the 3rd row (changing 3rd row)
3) Multiply 2nd row with 3 and add it to the 3rd row (changing 3rd row) Continued steps above.

Steps Continued...

4) Subtract row 3 from row 2
5) Multiply row 3 by $\frac{-3}{2}$ and add it to the 1st row (changing 1st row). Also, divide row 3 by 2.

Problem 2. *(25 points)*

(a) *(10 points)* Find the matrix of the linear transformation that first projects a vector $v \in \mathbb{R}^2$ on the line $y = x$, then rotates it counterclockwise by a $\pi/4$ angle, and then scales it by a factor $k > 0$.

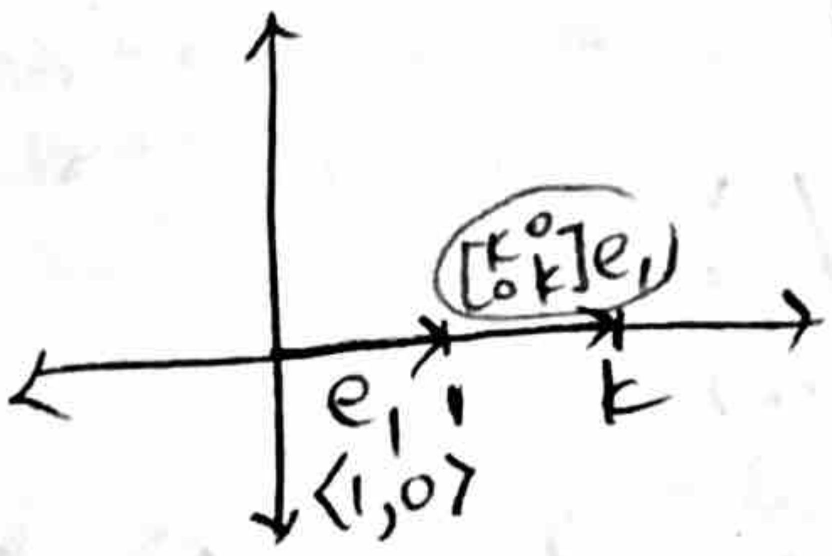
(b) *(15 points)* Let w be a vector on a line L in $\mathbb{R}^2$ that passes through the origin. Consider the $proj_L(v)$ which is the projection of another vector $v \in \mathbb{R}^2$ onto L. Finally let A be the 2×2 matrix whose columns are the vectors w and $proj_L(v)$. Is A invertible? Justify your answer.

(a) $S_k\left(Rot_{\frac{\pi}{4}}\left(proj_L(\vec{x})\right)\right) = \vec{y}$

$S_k = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}$

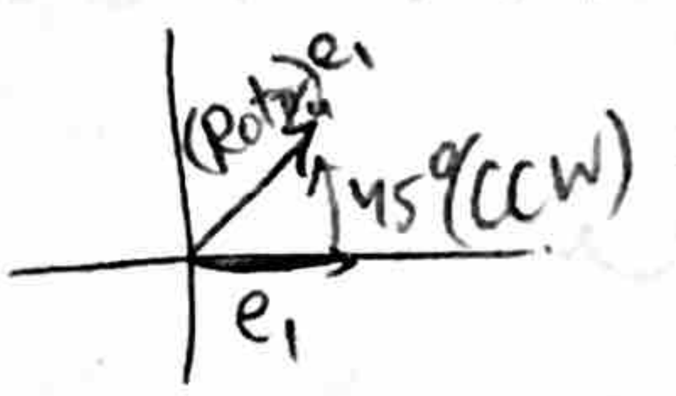
e_1 is scaled by a factor $k > 0$.



$\underbrace{\left(S_k Rot_{\frac{\pi}{4}} P_L\right)}\vec{x} = \vec{y}$

This is the matrix we want to find. (The composition of $S_k \circ Rot_{\frac{\pi}{4}} \circ P_L$

$Rot_{\frac{\pi}{4}} = \begin{bmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4}) \\ \sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix}$

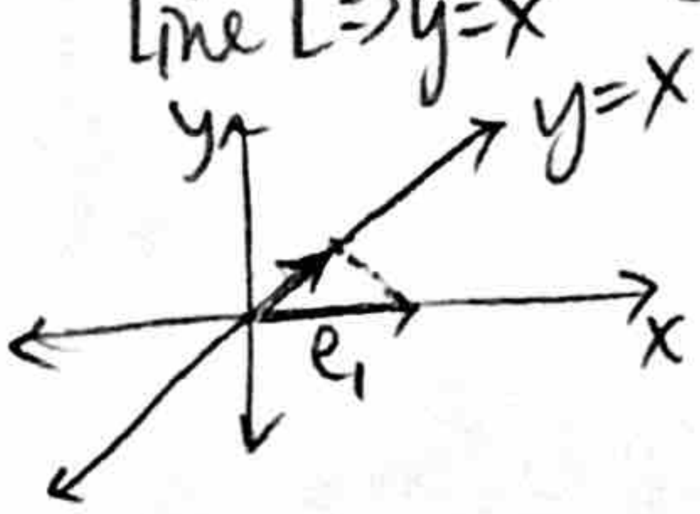


$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$\frac{\pi}{4} \cdot \frac{\cancel{180}\ 45}{\pi}$

$P_L = \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right)^2 & \left(\frac{1}{\sqrt{2}}\right)^2 \\ \left(\frac{1}{\sqrt{2}}\right)^2 & \left(\frac{1}{\sqrt{2}}\right)^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Line $L \Rightarrow y = x$



$\vec{u} = \langle 1, 1 \rangle \frac{1}{\sqrt{2}}$

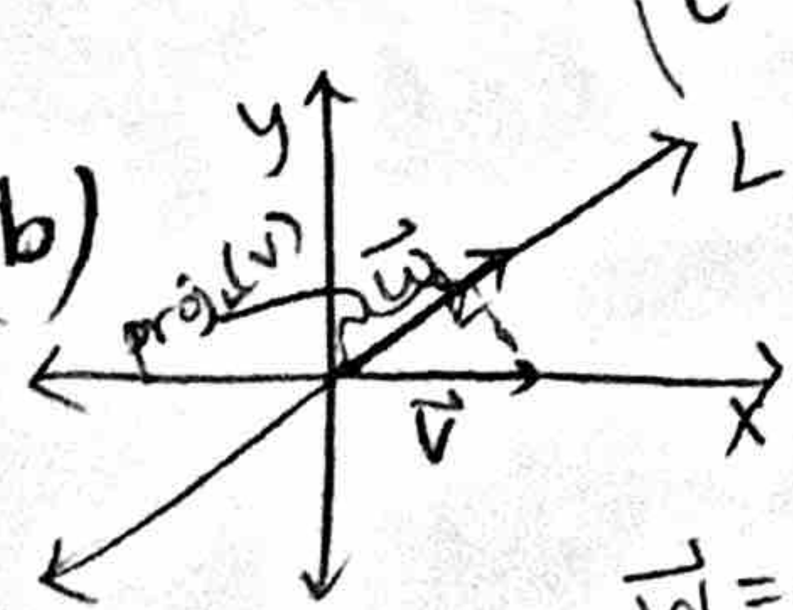
unit vector on L

$\underset{P_L}{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}} \underset{e_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ proj. of e_1 on L.

Answer

So, $\left(S_k Rot_{\frac{\pi}{4}}\right)P_L = \left(\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\right)\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{k}{\sqrt{2}} & -\frac{k}{\sqrt{2}} \\ \frac{k}{\sqrt{2}} & \frac{k}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \frac{k}{\sqrt{2}} & \frac{k}{\sqrt{2}} \end{bmatrix}$

(b)



$proj_L(\vec{v}) = \frac{1}{w_1^2 + w_2^2}\begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{w_1^2 v_1 + w_1 w_2 v_2}{w_1^2 + w_2^2} \\ \frac{w_1 w_2 v_1 + w_2^2 v_2}{w_1^2 + w_2^2} \end{bmatrix}$

$\vec{w} = \langle w_1, w_2 \rangle$

$\vec{v} = \langle v_1, v_2 \rangle$

So, $A = \begin{bmatrix} \frac{w_1^2 v_1 + w_1 w_2 v_2}{w_1^2 + w_2^2} & w_1 \\ \frac{w_1 w_2 v_1 + w_2^2 v_2}{w_1^2 + w_2^2} & w_2 \end{bmatrix}$

continued...

Because the $\det(A) \neq 0$, $\Rightarrow$ $w_2\left(\frac{w_1^2 v_1 + w_1 w_2 v_2}{w_1^2 + w_2^2}\right) - w_1\left(\frac{w_1 w_2 v_1 +}{w_1^2 + w_2^2}\right.$

A is invertible.

$\neq 0.$

Another explanation for why A is invertible because A is linearly indep. system since $\vec{w}$ and $\text{proj}_L \vec{v}$ are not linear combinations or multiples of each other. So, $\ker(A) = \{\vec{0}\}$ $\Rightarrow$ which means that A is invertible.

Doing this in full computational detail wasn't as efficient as it could've been.

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Problem 3. *(25 points)*

(a) *(10 points)* Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with the property that

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = \begin{bmatrix}9\\3\end{bmatrix} \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}2\\4\end{bmatrix}.$$

(b) *(15 points)* Let A be a given 3×3 matrix, and v be a given vector in $\mathbb{R}^3$. Is the following transformation $F : \mathbb{R}^3 \to \mathbb{R}^3$ that is given by

$$F(y) = v \times y + Ay \text{ for all } y \in \mathbb{R}^3,$$

where

$$v \times y = \begin{bmatrix} v_2y_3 - v_3y_2 \\ v_3y_1 - v_1y_3 \\ v_1y_2 - v_2y_1 \end{bmatrix} \text{ for } v = (v_1, v_2, v_3),\ y = (y_1, y_2, y_3),$$

a linear transformation?

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(a) $A\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}9\\3\end{bmatrix} \longrightarrow 3a + b = 9$, $3c + d = 3$

$A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix} \longrightarrow a + 2b = 2$, $c + 2d = 4$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$

6K

$$\begin{array}{cccc|c} a & b & c & d & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 3 & 1 & 0 & 0 & 9 \\ 0 & 0 & 3 & 1 & 3 \\ 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right] \downarrow \left[\begin{array}{cccc|c} 3 & 1 & 0 & 0 & 9 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & -5 & 0 & 0 & 3 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{3} & 0 & 0 & 3 \\ 0 & -5 & 0 & 0 & 3 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 1 & 2 & 4 \end{array}\right] \longrightarrow -5b = 3,\quad b = -\frac{3}{5}$$

$a + \frac{1}{3}b = 3$

$a + \frac{1}{3}\left(\frac{3}{5}\right) = 3$

$a - \frac{1}{5} = 3$

$a = 3\frac{1}{5}$

$$\left[\begin{array}{cccc|c} 1 & \frac{1}{3} & 0 & 0 & 3 \\ 0 & -5 & 0 & 0 & 3 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & -5 & -9 \end{array}\right]$$

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$-5d = -9$

$d = \frac{9}{5}$

$3c + d = 3$

$3c = 3 - \frac{9}{5}$

$c = \frac{3 - \frac{9}{5}}{3}$

So,

$$A = \begin{bmatrix} 3\frac{1}{5} & -\frac{3}{5} \\ \frac{\left(3 - \frac{9}{5}\right)}{3} & \frac{9}{5} \end{bmatrix}$$

Answer

Continued...

(b) L.T. are defined by:-

$$f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$$

$$f(k\vec{x}) = kf(\vec{x})$$

$$F(\vec{y}) = \vec{v}\times\vec{y} + A\vec{y} = \begin{bmatrix} v_2y_3 - v_3y_2 \\ v_3y_1 - v_1y_3 \\ v_1y_2 - v_2y_1 \end{bmatrix} + A\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$F(k\vec{y}) = k(\cancel{F(\vec{y})}) = k\begin{bmatrix} v_2y_3 - v_3y_2 \\ v_3y_1 - v_1y_3 \\ v_1y_2 - v_2y_1 \end{bmatrix} + A\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

✓ Holds

$F(\vec{y})$ is already a combination of two matrix linear
Why? combinations (L.T.) $A\vec{y}$ and $\vec{v}\times\vec{y}$, so $F(\vec{y})$ must be closed under addition as well.

So, $\boxed{F(\vec{y}) \text{ is a linear transformation.}}$

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Problem 4. *(25 points)*

(a) *(10 points)* Is the set

$$V = \{(x, y) \in \mathbb{R}^2 \mid x^2 = y^2\}$$

a subspace of $\mathbb{R}^2$?

(b) *(15 points)* Let V, W be two subspaces of $\mathbb{R}^n$. Show that

$$V \cap W = \{x \in \mathbb{R}^n \mid x \in V \text{ and } x \in W\}$$

is also a subspace of $\mathbb{R}^n$.

(a) V includes $\{\vec{0}\}$ ✓ 3

V is not empty set ✓

V ~~holds~~ closed under linear combinations ✓

$k^2x^2 = k^2y^2$ ✓ closed under scalar multiplication

$x^2 + g(x,y) = y^2 + g(x,y)$ ✓ closed under addition.

So, $\boxed{\text{V is a subspace of } \mathbb{R}^2}$

(b) If V ~~is~~ and W are subspaces of $\mathbb{R}^n$ they both satisfy the above 3 condition (part a). The intersection of V and W must ~~also~~ contain $\{\vec{0}\}$ as well, and thus not be an empty set. Also, since V and W are both closed under linear combinations, any solutions of V and W both $(V \cap W)$ must closed under linear Combinations (closed under addition and ~~...~~ scalar multiplications). So, $V \cap W$ is also a subspace of $\mathbb{R}^n$.

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