Math 32B-1 Yeliussizov. Midterm 2

Exam time: 6:00-7:30 PM, February 27, 2017

Discussion section: Christian 1A Tue, 1B Thu; Maldague 1C Tue; 1D Thu

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There are 5 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

P 1 (15)	P 2 (15)	P 3 (20)	P 4 (20)	P 5 (20)	Total (90 pt)
9	10	16	11	17	63

Problem 1. (15 points) Let D be the region enclosed by $y=x$, $y=3x$, $y=1-x$, $y=2-x$.
(a) (5 points) Find a map $F(x,y)$ whose image $F(D)$ is a rectangle (i.e., maps D to a rectangle)
(b) (10 points) Evaluate $\iint_D \frac{y(x+y)}{x^3}\,dxdy$ using change of variables from F.

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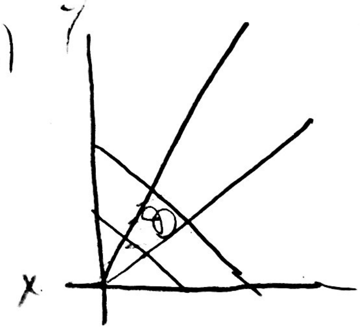
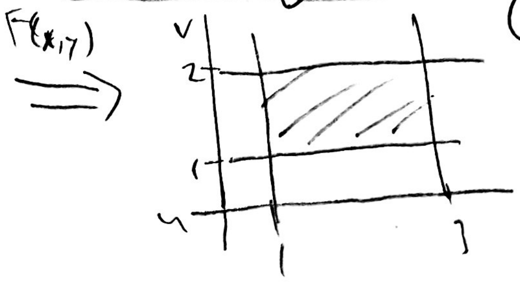
a)



$\boxed{F(x,y) = \left(\frac{y}{x}, y+x\right)}$

5

$2_y + 2_x = 2v$



b) $\mathrm{Jac}(F) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{-y}{x^2} & \frac{1}{x} \\ y+1 & x+1 \end{vmatrix} = \frac{-y}{x^2}(x+1) - \frac{y+1}{x}$

$= \frac{-y}{x} - \frac{y}{x^2} - \frac{y}{x} - \frac{1}{x} = \frac{-y}{x^2} - \frac{2y}{x} - \frac{1}{x}$

$= \frac{1}{x}\left(\frac{-y}{x} - 2y - 1\right)$

$= \frac{-1}{x^2}(y + 2yx + x)$

$\mathrm{Jac}(G) = \frac{1}{\mathrm{Jac}(F)} = \frac{x^2}{y+2yx+x}$

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$\iint_D \cancel{\frac{x^2}{y+2yx+x}} \cdot \frac{y(x+y)}{x^3}\,dxdy$

$\iint_D \frac{y(x+y)}{x(y+x+2yx)}\,dxdy$

$\cancel{\iint_{D} \ldots}$

Problem 2. (15 points) Let a, b, c be real constants. Show that $\operatorname{div}(\mathbf{F} \times \langle a, b, c\rangle) = 0$ if $\mathbf{F}$ is a conservative vector field.

let $F = \langle P, Q, R \rangle$ where P, Q, R are funcs. of $x, y,$ and/or z

$$F \times \langle a,b,c \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P & Q & R \\ a & b & c \end{vmatrix} = \hat{i}(cQ - bR) - \hat{j}(cP - aR) + \hat{k}(bP - aQ)$$

$$\operatorname{div}(F \times \langle a,b,c \rangle) = cQ - bR - cP + aR + bP - aQ =$$

for this to be zero, $cQ = cP$ $bR = bP$ $aR = aQ$

$Q = P$ $R = P$ $Q = R$

If F is conservative there exists a function f where $F = \nabla f$

$F = \langle f_x, f_y, f_z \rangle$

$F = \langle P, Q, R \rangle$

Has to be conservative due to Fundamental Thm of Conservative vector fields

10

16

Problem 3. (20 points) Consider the path C parametrized by $\mathbf{r}(t) = (\cos 2t, \sin 2t, t)$ for $0 \le t \le 1$.
(a) (10 points) Evaluate the length of C.
(b) (10 points) Evaluate the vector line integral $\int_C \mathbf{F}\, d\mathbf{r}$, where $\mathbf{F} = \langle -y, x, z \rangle$.

a) $r'(t) = (-2\sin(2t), 2\cos(2t), 1)$

$\|r'(t)\| = \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 1} = \sqrt{4+1} = \sqrt{5}$

10

$\text{Length} = \int_C 1\, ds = \int_0^1 1 \cdot \|r'(t)\|\, dt$

$= \int_0^1 \sqrt{5} = \sqrt{5}\,[t]_0^1 = \boxed{\sqrt{5}}$

b) $F = \langle -\sin(2t), \cos(2t), t \rangle$

$\int_C F\, dr = \int_0^1 F \cdot r'(t)\, dt$

need derivatin!

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$= \int_0^1 \langle -\sin(2t), \cos(2t), t \rangle \cdot \langle \cos(2t), \sin(2t), t \rangle\, dt$

$= \int_0^1 \left(-\sin(2t)\cos(2t) + \sin(2t)\cos(2t) + t^2\right) dt$

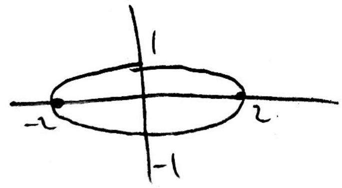
$= \int_0^1 t^2\, dt$

$= \frac{1}{3}[t^3]_0^1 = \cancel{\boxed{\frac{1}{3}}}$

Problem 4. (20 points) Let C be a path from $(2,0)$ to $(0,1)$ along the ellipse $x^2+4y^2=4$ in the first quadrant, oriented counterclockwise. $\langle y - y\sin x,\ x + \cos x\rangle$

(a) (10 points) Let $\mathbf{F} = \cancel{\langle -y\sin x, x+\cos x\rangle}$. Show that $\mathbf{F}$ is conservative, find a potential function $f(x,y)$ so that $\mathbf{F} = \nabla f$, and evaluate $\int_C \mathbf{F}\, d\mathbf{r}$.

(b) (10 points) Let $\mathbf{F} = \langle -y, x\rangle$. Is it conservative? Evaluate $\int_C \mathbf{F}\, d\mathbf{r}$.



a) $F = \langle P, Q\rangle = \langle y - y\sin x,\ x + \cos x\rangle$

$\text{curl}(\vec{F}) = Q_x - P_y = (1-\sin x) - (1 - \sin x) = 0$

Since D is simply connected (i.e., no "holes") and $\text{curl}(\vec{F}) = 0$, F is conservative

$\frac{\partial f}{\partial x} = y - y\sin x$ $\qquad$ $\frac{\partial f}{\partial y} = x + \cos(x)$

$f = yx + y\cos x + C(y)$ $\qquad$ $x + \cos x + \frac{\partial C}{\partial y} = x + \cos(x)$

$\frac{\partial C}{\partial y} = 0 \Rightarrow C(y) = 0$

$\boxed{f(x,y) = yx + y\cos(x) + C \text{ where } C \text{ is a constant real number}}$

$r(t) = (2\cos t, \sin(t))$

$r'(t) = \langle -2\sin t, \cos(t)\rangle$ $\quad \|r'(t)\| = \sqrt{4\sin^2 t + \cos^2 t} = $?

?

b) $\text{curl}(\vec{F}) = Q_x - P_y = 1 - (-1) = 2$

Not conservative since $\text{curl}(\vec{F}) \neq 0$ even though domain is simply connected

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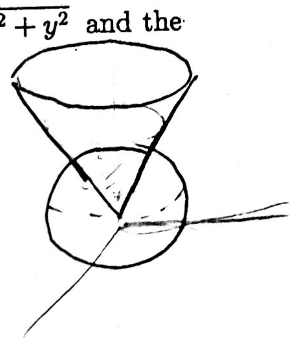
Problem 5. (20 points) Compute the area of the surface enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=1$ from above.



$G(z,\theta) = (z\cos\theta, z\sin\theta, z)$

$T_\theta = \langle -z\sin\theta, z\cos\theta, 0\rangle$

$T_z = \langle \cos\theta, \sin\theta, 1\rangle$

$$T_\theta \times T_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -z\sin\theta & z\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = \hat{i}(z\cos\theta) - \hat{j}(-z\sin\theta) + \hat{k}(-z\sin^2\theta - z\cos^2\theta)$$
$$= \hat{i}(z\cos\theta) - \hat{j}(-z\sin\theta) + \hat{k}(-z)$$

$$\|T_\theta \times T_z\| = \sqrt{z^2\cos^2\theta + z^2\sin^2\theta + z^2} = z\sqrt{1+1} = z\sqrt{2}$$

← cone

$$AREA = \iint 1 \cdot \|T_\theta \times T_z\| \, dA = \iint z\sqrt{2}\, dA$$
$$= \int_0^{2\pi}\int_0 \cancel{z\sqrt{2}\, dz\, d\theta} = ?$$

$G(\theta,\phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$T_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)$

$T_\phi = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)$

$$T_\theta \times T_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{vmatrix}$$
$$= \hat{i}(-\sin^2\phi\cos\theta) - \hat{j}(\sin^2\phi\sin\theta) + \hat{k}(-\sin\phi\cos\phi\sin^2\theta - \sin\phi\cos\phi\cos^2\theta)$$
$$= \hat{i}(\quad) - \hat{j}(\quad) + \hat{k}(-\sin\phi\cos\phi)$$

$$\|T_\theta \times T_\phi\| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$$
$$= \sin\phi\sqrt{\sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi}$$
$$= \sin\phi \quad \checkmark$$

$$AREA = \iint 1 \cdot \|T_\theta \times T_\phi\| \, dA = \int_0^{2\pi}\int_0^{\pi/4} \sin\phi \, d\phi\, d\theta$$
$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/4} \sin\phi\, d\phi$$
$$= [\theta]_0^{2\pi} \cdot [-\cos\phi]_0^{\pi/4}$$
$$= 2\pi\left(-\frac{\sqrt{2}}{2} - (-1)\right) = 2\pi\left(-\frac{\sqrt{2}}{2} + 1\right)$$
$$\boxed{= 2\pi - \pi\sqrt{2}} \quad \checkmark$$

cone?