Math 32B-1 Yeliussizov. Midterm 1

Exam time: 6:00-7:30 PM, January 30, 2017

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Discussion section: Christian 1A Tue, 1B Thu; Maldague 1C Tue, 1D Thu

There are 5 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

P 1 (15)	P 2 (15)	P 3 (20)	P 4 (20)	P 5 (20)	Total (90 pt)
9	15	20	20	17	81

Problem 1. (15 points) Evaluate the double integral over the given rectangular domain in the xy-plane

$$\iint_R (1+y+xe^{xy})dA, \qquad R=[0,2]\times[-1,1].$$

$$\int_0^2\int_{-1}^1 (1+y+xe^{xy})\,dy\,dx$$

$$\int^2 \left[y+\frac{y^2}{2}+e^{xy}\right]_{-1}^{1} dx$$

$$\int_0^2 \left(1+\frac{1}{2}+e^{x}\right)dx - \left(-1+\frac{1}{2}+e^{-x}\right)$$

$$\int_0^2 \frac{3}{2}+e^{x}\,dx$$

$$\left[\frac{3}{2}x+e^{x}\right]_0^2$$

$$3+e^2-(0+1)$$

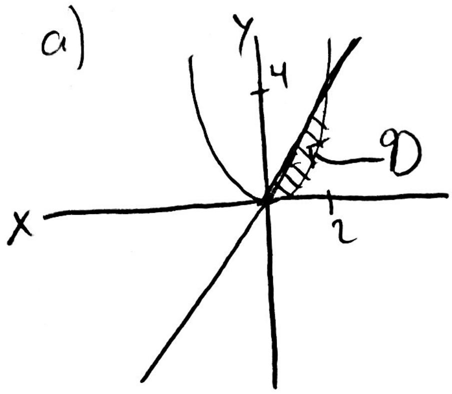
$$3+e^2-1$$

$$\boxed{2+e^2}$$

9/15

Problem 2. (15 points) Let D be the region bounded by $y = 2x$ and $y = x^2$.
(a) (5 points) Sketch the region D in the xy-plane.
(b) (10 points) Compute the double integral of $f(x, y) = \dfrac{x}{4-y}$ over the domain D. (Choose the order of integration that enables you to evaluate the integral.)

a)



b) $2x = x^2$

$0 = x^2 - 2x$

$0 = x(x-2)$

$x = 0, 2$

$y = 2x \Rightarrow x = \frac{y}{2}$

$y = x^2 \Rightarrow x = \sqrt{y}$

$$\int_0^4 \int_{y/2}^{\sqrt{y}} \frac{x}{4-y}\, dx\, dy$$

$$\frac{1}{2}\int_0^4 \frac{1}{4-y}\left[x^2\right]_{\frac{y}{2}}^{\sqrt{y}} dy$$

$$\frac{1}{2}\int_0^4 \frac{1}{4-y}\left(y - \frac{y^2}{4}\right) dy$$

$$\frac{1}{2}\int_0^4 \frac{1}{4}\,\frac{y(4-y)}{4-y}\, dy$$

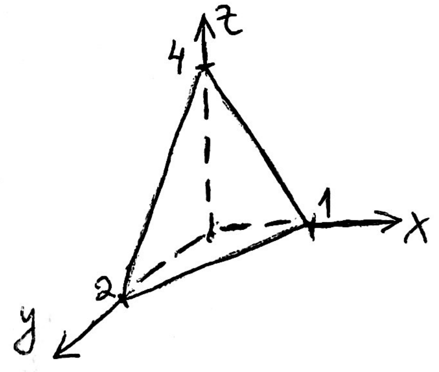
$$\frac{1}{8}\int_0^4 y\, dy$$

$$\frac{1}{2}\left(\frac{1}{8}\right)\left[y^2\right]_0^4$$

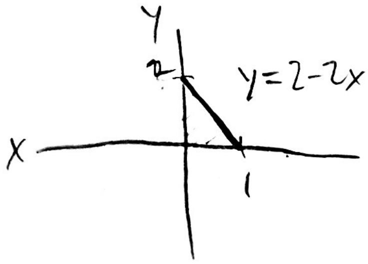
$$\frac{1}{16}(16-0) = \boxed{1}$$

15

Problem 3. (20 points) Let W be the tetrahedron in the first octant $x, y, z \geq 0$ with vertices at the points $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,4)$ (see the figure). Evaluate the triple integral of the function $f(x,y,z) = 1/(1-x)$ over W.



$z = 4 - 2y - 4x$



$$\int_0^1 \int_0^{2-2x} \int_0^{4-2y-4x} \frac{1}{(1-x)}\, dz\, dy\, dx$$

$$\int_0^1 \int_0^{2-2x} \frac{1}{1-x} \left[z\right]_0^{4-2y-4x} dy\, dx$$

$$\int_0^1 \int_0^{2-2x} \frac{1}{1-x} (4-2y-4x)\, dy\, dx$$

$$\int_0^1 \frac{1}{1-x} \left[4y - y^2 - 4xy\right]_0^{2-2x} dx$$

$$\int_0^1 \left(\frac{1}{1-x}\left(8-8x-(2-2x)^2-4x(2-2x)\right)\right) dx$$

$$\int_0^1 \frac{1}{1-x} (8-8x-(4-8x+4x^2)-8x+8x^2\, dx$$

$$\int_0^1 \frac{1}{1-x} (4-4x^2-8x+8x^2)\, dx$$

$$\int_0^1 \frac{1}{1-x} (4x^2-8x+4)\, dx$$

$$4\int_0^1 \frac{1}{1-x} (x^2-2x+1)\, dx$$

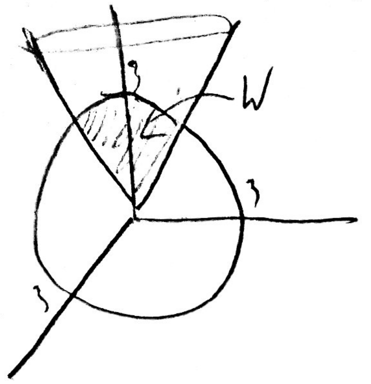
$$4\int_0^1 \frac{1}{1-x} (x-1)^2\, dx$$

$$-4\int_0^1 \frac{1}{x-1} (x-1)^2\, dx$$

$$-4\int_0^1 x-1\, dx$$

$$-4\left[\frac{x^2}{2} - x\right]_0^1 = -4\left(\frac{1}{2}-1\right) = -4\left(\frac{-1}{2}\right) = \boxed{2}$$

Problem 4. (20 points) Let W be the region bounded by the sphere $x^2+y^2+z^2=9$ and (above) the cone $z=\sqrt{x^2+y^2}$. Find the volume of W using spherical coordinates.



$x^2+y^2+z^2=9 \Rightarrow \rho^2=9 \Rightarrow \rho=3$

$z=\sqrt{x^2+y^2} \Rightarrow \rho\cos\varphi=\sqrt{}$

$\int_0^{2\pi}\int_0^{\pi/4}\int_0^3 1\cdot\rho^2\sin\varphi \, d\rho \, d\varphi \, d\theta$

$\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\varphi\left[\rho^3\right]_0^3 d\varphi \, d\theta$

$\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\varphi(27)\, d\varphi \, d\theta$

$9\int_0^{2\pi}\int_0^{\pi/4}\sin\varphi \, d\varphi \, d\theta$

$-9\int_0^{2\pi}\left[\cos\varphi\right]_0^{\pi/4} d\theta$

$-9\int_0^{2\pi}\frac{1}{\sqrt{2}}-1 \, d\theta$

$-9\int_0^{2\pi}\frac{1-\sqrt{2}}{\sqrt{2}} \, d\theta$

$\frac{-9(1-\sqrt{2})}{\sqrt{2}}\left[\theta\right]_0^{2\pi}$

$\frac{-18\pi(1-\sqrt{2})}{\sqrt{2}}$

$\boxed{\mathrm{Vol}(W)=\frac{18\pi(\sqrt{2}-1)}{\sqrt{2}}}$

$x^2+y^2+x^2+y^2=9$

$2(x^2+y^2)=9$

$x^2+y^2=\frac{9}{2}$

$(\rho\sin\varphi\cos\theta)^2+(\rho\sin\varphi\sin\theta)^2=\frac{9}{2}$

$\rho^2\sin^2\varphi\cos^2\theta+\rho^2\sin^2\varphi\sin^2\theta=\frac{9}{2}$

$\rho^2\sin^2\varphi=\frac{9}{2}$

Since its on intersection, $\rho=3$

$9\sin^2\varphi=\frac{9}{2}$

$\sin^2\varphi=\frac{1}{2}$

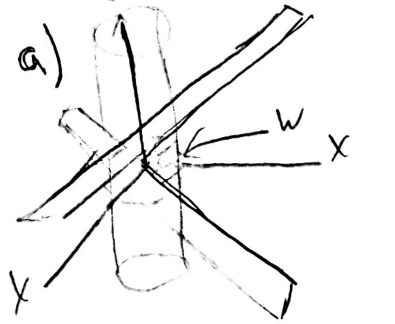
$\sin\varphi=\frac{1}{\sqrt{2}}$

$\varphi=\frac{\pi}{4}$

20

Problem 5. (20 points) Let W be the region bounded by the cylinder $x^2+y^2=1$ and two half-planes $x=|z|$.
(a) (10 points) Find the volume of W.
(b) (10 points) Find the centroid of W (i.e. the center of mass assuming the mass density $\delta(x,y,z)=1$) using cylindrical coordinates. Cylindrical:

a)

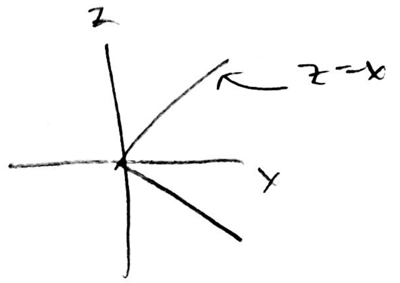


$z=x \Rightarrow z=r\cos\theta$

Due to symmetry, you can do top half + multiply by 2

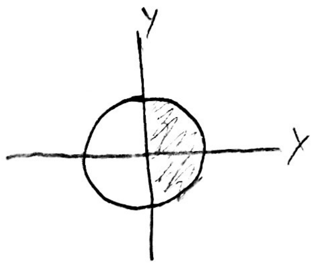
$$2\int_{-\pi/2}^{\pi/2}\int_0^1\int_0^{r\cos\theta} 1\cdot r\,dz\,dr\,d\theta$$

$$2\int_{-\pi/2}^{\pi/2}\int_0^1 r\,[z]_0^{r\cos\theta}\,dr\,d\theta$$



$$2\int_{-\pi/2}^{\pi/2}\int_0^1 r^2\cos\theta\,dr\,d\theta$$

$$\frac{2}{3}\int_{-\pi/2}^{\pi/2}\cos\theta\,[r^3]_0^1\,d\theta$$



$$\frac{2}{3}\int_{-\pi/2}^{\pi/2}\cos(\theta)\,d\theta=\frac{2}{3}[\sin\theta]_{-\pi/2}^{\pi/2}=\frac{2}{3}(1-(-1))=\frac{4}{3}$$

$$\boxed{Vol(W)=\frac{4}{3}}$$

b) $$y_{cm}=\frac{\iiint_W (r\sin\theta)\,dW}{\iiint_W dW}=\frac{0}{\iiint_W dW}=0$$ since $f(x,y,z)=y$ for numerator to find y_{cm}, and $f(x,-y,z)=-y=-f(x,y,z)$. Since range of integration is centered at $y=0$ + $f(x,y,z)=y$ is symmetric about $y=0$, $y_{cm}=0$

$$z_{cm}=\frac{\iiint_W zr\,dW}{\iiint_W dW}=\frac{0}{\iiint_W dW}=0$$ since $f(x,y,z)=z$ for numerator to find y_{cm}, and $f(x,y,-z)=-z=-f(x,y,z)$. Since range of integration is centered at $z=0$ + $f(x,y,z)=z$ is symmetric about $z=0$, $z_{cm}=0$.

$\boxed{\text{* } x_{cm} \text{ on back!}}$

b (cont) $2\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^1\int_0^{r\cos\theta} r^2\cos\theta\, z\, dz\, dr\, d\theta$

$$2\int_{-\frac{\pi}{2}}^{\pi/2}\int_0^1 r^2\cos\theta\left[z\right]_0^{r\cos\theta} dr\, d\theta$$

$$2\int_{-\frac{\pi}{2}}^{\pi/2}\cos^2\theta\int_0^1 r^3\, dr\, d\theta$$

$$\frac{2}{4}\int_{-\frac{\pi}{2}}^{\pi/2}\cos^2\theta\left[r^4\right]_0^1 d\theta$$

$$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\pi/2}\cos^2\theta\, d\theta$$

?

$$\frac{1}{2}\left[\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$\frac{1}{2}(1-(-1)) = 1$$

$$X_{cm} = \frac{\iiint_W r\cos\theta\, dW}{\iiint_W dW} = \frac{1}{\frac{4}{3}} = \frac{3}{4}$$

$\boxed{\text{Centroid of } W \text{ is at } \left(\frac{3}{4}, 0, 0\right)}$