33B Visan | Midterm 2 | November 4th

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Section: 1E

$$= \begin{cases} 1A & \text{Tuesday with J. Yu} \\ 1B & \text{Thursday with J. Yu} \\ 1C & \text{Tuesday with D. Lichko} \\ 1D & \text{Thursday with D. Lichko} \\ 1E & \text{Tuesday with A. Lin} \\ 1F & \text{Thursday with A. Lin} \end{cases}$$

Rules:

- There are **FOUR** problems for a total of 50 points.
- Use the backs of the pages.
- No calculators, computers, notes, books, e.t.c..
- Out of consideration for your classmates, no chewing, humming, pen-twirling, snoring, e.t.c.. Try to sit still.
- Turn off your cell-phone.

1	2	3	4	Σ
10	10	13	14	47

(1) (12 points) Consider the equation

$$x''' + x'' + 4x' + 4x = 0.$$

(a) Find the general solution to this equation.

(b) Find the solution to the equation satisfying the initial conditions

$$x(0) = 1, \quad x'(0) = -1, \quad \text{and} \quad x''(0) = \alpha,$$

where α denotes a real constant.

(c) Find the value of the constant $\alpha \in (-\infty, \infty)$ for which the solution you found in part (b) approaches zero as $t \to \infty$.

10

a) $x''' + x'' + 4x' + 4x = 0$

charact. equation

$\lambda^3 + \lambda^2 + 4\lambda + 4 = 0$

$\lambda^2(\lambda+1) + 4(\lambda+1) = 0$

$(\lambda^2+4)(\lambda+1) = 0$

charact. roots

$\lambda_1 = -1$

$\lambda_2 = 2i$

$\lambda_3 = -2i$

fund. set of solutions

$\phi_1 = e^{-t}$

$\phi_2 = \cos(2t)$

$\phi_3 = \sin(2t)$

General solution

$\boxed{x(t) = c_1 e^{-t} + c_2\cos(2t) + c_3\sin(2t)}$

b) $x(t) = c_1 e^{-t} + c_2\cos(2t) + c_3\sin(2t)$

$x'(t) = -c_1 e^{-t} - 2c_2\sin(2t) + 2c_3\cos(2t)$

$x''(t) = c_1 e^{-t} - 4c_2\cos(2t) - 4c_3\sin(2t)$

plug in initial cond.

$1 = c_1 \cdot 1 + c_2 \cdot 1 + \cancel{c_3 \cdot 0}$

$-1 = -c_1 \cdot 1 - \cancel{2c_2 \cdot 0} + 2c_3 \cdot 1$

$\alpha = c_1 \cdot 1 - 4c_2 \cdot 1 - \cancel{4c_3 \cdot 0}$

$\Downarrow$

$$\begin{cases} c_1 + c_2 = 1 \\ -c_1 + 2c_3 = -1 \\ c_1 - 4c_2 = \alpha \end{cases}$$

$\therefore c_2 = \dfrac{1-\alpha}{5}$

$c_1 = \dfrac{5-(1-\alpha)}{5} = \dfrac{4+\alpha}{5}$

$c_3 = \dfrac{\alpha - 1}{10}$

Solution

$$\boxed{x(t) = \left(\frac{4+\alpha}{5}\right)e^{-t} + \left(\frac{1-\alpha}{5}\right)\cos(2t) + \left(\frac{\alpha-1}{10}\right)\sin(2t)}$$

c) $\lim_{t\to\infty} x(t) = \left(\dfrac{4+\alpha}{5}\right)e^{-t}$

let $\boxed{\alpha = -4}$

2

(2) (10 points) Find the general solution to the differential equation

$$x'' - 5x' + 4x = \sin(t) + 8.$$

Assoc. Homo. eqs.

Charact. eq.

$\lambda^2 - 5\lambda + 4 = 0$

$(\lambda - 4)(\lambda - 1) = 0$

charact. roots

$\lambda_1 = 4$

$\lambda_2 = 1$

Fund set of solutions

$\phi_1 = e^{4t}, \ \phi_2 = e^t$

Find particular solution, x_p

Method of undet coeff.

try $x_p = A\sin t + B\cos t + C$

$x_p' = A\cos t - B\sin t$

$x_p'' = -A\sin t - B\cos t$

plug into inhom eq.

$-A\sin t - B\cos t - 5(A\cos t - B\sin t) + 4(A\sin t + B\cos t + C) = \sin(t) + 8$

$(3A + 5B)\sin(t) + (3B - 5A)\cos(t) + 4C = \sin(t) + 8$

$$\begin{cases} 3A + 5B = 1 \\ 3B - 5A = 0 \\ 4C = 8 \end{cases} \Rightarrow \begin{matrix} A = 3/34 \\ B = 5/34 \\ C = 2 \end{matrix}$$

$\therefore x_p = 3/34 \sin t + 5/34 \cos t + 2$

General solution

$x(t) = 3/34 \sin t + 5/34 \cos t + 2 + c_1 e^{4t} + c_2 e^t$

(3) (14 points)

Consider the equation

$$tx'' + 2x' + tx = 1 \quad \text{with} \quad t > 0.$$

(a) Verify that $\phi_1(t) = \frac{\sin(t)}{t}$ and $\phi_2(t) = \frac{\cos(t)}{t}$ form a fundamental set of solutions to the associated homogeneous equation for $t \in (0, \infty)$.

(b) Find a particular solution to the given inhomogeneous equation.

(c) Write down the general solution to the inhomogeneous equation.

a) Assoc. homo. eq.

$$tx'' + 2x' + tx = 0$$

plug ϕ_1 & ϕ_2 into eq.

+2

$$t\left(\frac{\sin t}{t}\right) + 2\left(\frac{\cos t}{t} - \frac{\sin t}{t^2}\right) + t\left(\frac{-\sin t}{t} - \frac{\cos t}{t^2} - \frac{\cos t}{t^2} + \frac{2\sin t}{t^3}\right) = 0$$

$0 = 0$ ✓

ϕ_1 is a solution to assoc homo. eq.

+2

$$t\left(\frac{\cos t}{t}\right) + 2\left(\frac{-\sin t}{t} - \frac{\cos t}{t^2}\right) + t\left(\frac{-\cos t}{t} + \frac{\sin t}{t^2} + \frac{\sin t}{t^2} + \frac{2\cos t}{t^3}\right) = 0$$

$0 = 0$ ✓

ϕ_2 is a solution to assoc. homo. eq.

Verify ϕ_1 & ϕ_2 are lin. indep.

$$W(\phi_1, \phi_2)(t) = \det\begin{pmatrix} \frac{\sin(t)}{t} & \frac{\cos(t)}{t} \\ \frac{\cos t}{t} - \frac{\sin t}{t^2} & \frac{-\sin t}{t} - \frac{\cos t}{t^2} \end{pmatrix}$$

$$= \frac{-\sin^2 t}{t^2} - \frac{\sin(t)\cos(t)}{t^3} - \frac{\cos^2(t)}{t^2} + \frac{\sin(t)\cos(t)}{t^3}$$

$$= \frac{-(\sin^2 t + \cos^2 t)}{t^2}$$

$$= -1/t^2 \neq 0 \text{ at } t \in (0, \infty)$$

+2 Because ϕ_1 & ϕ_2 are lin indep from $t \in (0,\infty)$ & are both solutions, they form a fund. set of solutions

b) Find particular solution

Variation of Parameters

$$x_p = u_1\phi_1 + u_2\phi_2$$

$$\therefore \begin{cases} u_1'\phi_1 + u_2'\phi_2 = 0 \\ u_1'\phi_1' + u_2'\phi_2' = f(t) \end{cases}$$

from given inhomogen. eq: $f(t) = 1/t$

$$u_1 = -\int \frac{f(t)\phi_2(t)\,dt}{W(\phi_1,\phi_2)(t)} = +\int \frac{\cancel{t^2}\cos(t)\,dt}{\cancel{t^2}}$$

$$= +\sin(t) + C_1$$

$$u_2 = \int \frac{f(t)\phi_1(t)\,dt}{W(\phi_1,\phi_2)(t)} = -\int \frac{\cancel{t^2}\sin(t)}{\cancel{t^2}}\,dt$$

$$= +\cos(t) + C_2$$

disregard c_1, c_2 b/c the form fund. set of sol. for assoc. homo. eq.

$$x_p = \frac{-\sin^2(t)}{t} + \frac{-\cos^2 t}{t}$$

+5

$$\boxed{x_p = \frac{+1}{t}}$$

c) General solution

$$x(t) = x_p + c_1\phi_1 + c_2\phi_2$$

+2

$$\boxed{x(t) = \frac{+1}{t} + c_1\frac{\sin t}{t} + c_2\frac{\cos t}{t}}$$

(4) (14 points)

Consider the differential equation

$$t^2x'' - 2x = t^2 \quad \text{with} \quad t > 0. \tag{1}$$

(a) Verify that $\phi_1(t) = t^2$ is a solution to the associated homogeneous equation

$$t^2x'' - 2x = 0. \tag{2}$$

(b) Look for a solution to the inhomogeneous equation (1) of the form $x(t) = v(t)\phi_1(t)$. Plug this into equation (1) and derive a differential equation for v. Solve it.

(c) Write down the general solution to equation (1).

(d) Write down a fundamental set of solutions for equation (2) for $t > 0$.

a) plug $\phi_1(t)$ into $t^2\phi_1'' - 2\phi_1 = 0$

$t^2(2) - 2(t^2) = 0$

$0 = 0$ ✓

∴ ϕ_1 is a solution to associated homo. eq. ✓

b) plug $x(t) = v(t)\phi_1(t)$ into given eq (1)

$t^2(v''\phi_1 + 2v'\phi_1' + v\phi_1'') - 2(v\phi_1) = t^2$

$t^2\phi_1 v'' + 2t^2\phi_1' v' + (t^2\phi_1'' - 2\phi_1)v = t^2$

$t^4v'' + 4t^3v' + \cancel{(t^2 \cdot 2 - 2 \cdot t^2)v} = t^2$

$t^4v'' + 4t^3v' = t^2$

$u = v'$

$t^4u' + 4t^3u = t^2$

$u' + 4t^{-1}u = t^{-2}$ ✓

$f(t) = -4t^{-1} \qquad g(t) = t^{-2}$

$a(t) = e^{+\int 4t^{-1}dt} = e^{4\ln(t)} = t^4$

$u(t) = \frac{1}{a(t)}\int a(t)g(t)\,dt$

$= t^{-4}\int t^2\,dt$

$= \frac{t^{-4}t^3}{3} + ct^{-4}$

$= \frac{1}{3t} + ct^{-4}$

$v(t) = \int \frac{1}{3t} + ct^{-4}$

$= \frac{1}{3}\ln|t| - \frac{c}{3t^3} + d$

$$\boxed{x(t) = \left(\tfrac{1}{3}\ln|t|\right)t^2}$$

c) general solution

$$\boxed{x(t) = \left(\tfrac{1}{3}\ln|t| - \tfrac{c}{3t^3} + d\right)t^2}$$

d) fund. set of solutions

$$\boxed{\phi_2 = \frac{-1 \cdot t^2}{3t^3} \quad \phi_1 = t^2}$$