Midterm 2

Name: [illegible]

UID: [illegible]

Section:

Tuesday:	Thursday:	
1A	1B	TA: Benjamin Johnsrude
1C	1D	TA: Alexander Kastner
1E	1F	TA: Max Zhou

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. You may not use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You **must show all your work** to receive credit. Simplify your answers as much as possible.

You may use the front and back of the page for your answers. Do not write answers for one question on the page of another question.

Please do not write below this line on this page.

Question	Points	Score
1	10	
2	10	
3	12	
4	8	
Total:	40	40

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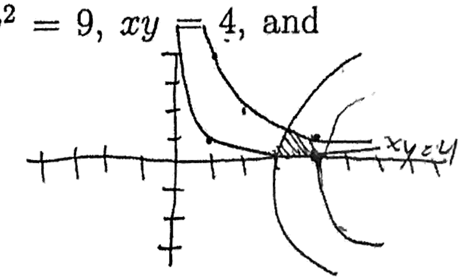
1. (10 points) Calculate the integral $\iint_D \frac{x^2+y^2}{x^2-y^2}\,dx\,dy$ where $\mathcal{D}$ is the region in the first quadrant enclosed by the curves $x^2-y^2=16$, $x^2-y^2=9$, $xy=4$, and $xy=1$.

one-sheet hyperbola

$u = x^2 - y^2, \quad v = xy \Rightarrow \frac{v}{x} = y$

$u = x^2 - \frac{v^2}{x^2}$



$\text{Jac}^{-1}(G) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2x^2 + 2y^2$

$9 \le u \le 16$

$1 \le v \le 4$

$\text{Jac}(G) = \frac{1}{2(x^2+y^2)}$

$\int_{v=1}^{4} \int_{u=9}^{16} \frac{x^2+y^2}{u} \left|\text{Jac}(G)\right| du\,dv$

$= \int_1^4 \int_9^{16} \frac{x^2+y^2}{u} \left(\frac{1}{2(x^2+y^2)}\right) du\,dv$

$= \int_1^4 \int_9^{16} \frac{1}{2u}\, du\, dv$

$= \int_1^4 \frac{1}{2} \ln|u| \Big]_9^{16} dv = \int_{v=1}^{4} \ln|16|^{1/2} - \ln(9)^{1/2} dv$

$= \int_1^4 \ln|4| - \ln|3|\, dv$

$= \boxed{3\ln\left|\frac{4}{3}\right|}$ or $3\ln\frac{4}{3}$

$\ln\left(\frac{4}{3}\right)^3$

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2. Let $\mathbf{F}(x,y,z) = \langle \cos z, 2y, -x\sin z\rangle$, and let C be the curve parametrized by $\mathbf{r}(t) = \langle t^2+1, te^{t^2-t}, \pi t^5\rangle$ for $0 \le t \le 1$.

(a) (4 points) Calculate div($\mathbf{F}$) and curl($\mathbf{F}$).

(b) (2 points) Show that $\mathbf{F}$ is conservative without calculating a potential function.

(c) (2 points) Find a function f such that $\mathbf{F} = \nabla f$.

(d) (2 points) Calculate $\int_C \mathbf{F}\cdot d\mathbf{r}$.

a. $\operatorname{div}(F) = \nabla\cdot F = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle F_1, F_2, F_3\rangle = \frac{\partial}{\partial x}\cos z + \frac{\partial}{\partial y}2y - \frac{\partial}{\partial z}x\sin z = 0 + \boxed{2 - x\cos z}$

$\operatorname{curl}(F) = \nabla\times\nabla f = \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos z & 2y & -x\sin z\end{vmatrix} = i\left[\frac{\partial}{\partial y}(-x\sin z) - \frac{\partial}{\partial z}(2y)\right] - j\left[\frac{\partial}{\partial x}(-x\sin z) - \frac{\partial}{\partial z}(\cos z)\right] + k\left[\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(\cos z)\right]$

$0 - 0$, $-\sin z + \sin z = 0$, 0, 0

$\operatorname{curl}(F) = \vec{0} = \boxed{\langle 0,0,0\rangle}$

$= -j[-\sin z + \sin z] = \boxed{\cancel{\langle 0, \sin z, 0\rangle}}$

$= \vec{0}$

b. curl(F) $= \vec{0}$ and cross partial equality is satisfied. (in other words) and F is simply connected domain.

$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ $\quad 0 = 0$ ✓

$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$ $\quad 0 = 0$ ✓

$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$ $\quad -\sin z = -\sin z$ ✓

curl($\vec{F}$) $= \vec{0}$ or cross-partials match up.
+ Simply connected domain.
Since F is defined in all of R^3 space.

c. $\int \frac{\partial F_1}{\partial x}\,dx = x\cos z + g(y,z)$

$\int \frac{\partial F_2}{\partial y}\,dy = y^2 + k(x,z)$

$\int \frac{\partial F_3}{\partial z}\,dz = x\cos z + h(x,y)$

$\boxed{f = x\cos z + y^2 + C'}$

d. $\int_C F\cdot dr$ $\quad r'(t) = \langle 2t, e^{t^2-1} + [2t-1]te^{t^2-1}, 5\pi t^4\rangle$

$= \int_C F(r(t))\cdot r'(t)\,dt$

$r(1) = \langle 2, 1, \pi\rangle$

$r(0) = \langle 1, 0, 0\rangle$

Since it's conservative, it's path independent and

$\int_C F\cdot dr = f(r_2(t)) - f(r_1(t))$

$= f(r(1)) - f(r(0))$

$= [2\cos\pi + 1] - [1\cos 0 + 0]$

$= -2 + 1 - 1 = \boxed{-2}$

111 1001

11 1101

3. (a) (5 points) Let C_0 be the curve given by $y = \dfrac{x^2}{2}$ for $0 \leq x \leq 1$ in the plane $z = 5$. Calculate $\displaystyle\int_{C_0} xz\, ds$.

$0 \leq y \leq \frac{1}{2}$

(b) (5 points) Suppose C is the curve with positive parametrization $\mathbf{r}(t) = \langle \cos t, 1, \sin t\rangle$ for $0 \leq t \leq 2\pi$. Calculate the work done by the force field $\mathbf{F} = \langle -z, xy, x\rangle$ in moving a particle along C.

(c) (2 points) Suppose $\mathbf{G}$ is a conservative vector field and C is the curve from part (b). Do we have enough information to calculate $\displaystyle\int_C \mathbf{G}\cdot d\mathbf{r}$? If yes, justify and evaluate the integral. If not, explain what information is lacking.

a. $r(t) = \langle t, \frac{t^2}{2}, 5\rangle \quad 0 \leq t \leq 1$

$r'(t) = \langle 1, t, 0\rangle$

$\|r'(t)\| = \sqrt{1+t^2}$

$\int F(r(t))\,\|r'(t)\|\,dt$

$= \int_0^1 t(5)\sqrt{1+t^2}\,dt$

$= 5\int_0^1 t\sqrt{1+t^2}\,dt$

$u = 1+t^2$

$du = 2t\,dt$

$= \frac{5}{2}\int_1^2 u^{1/2}\,du$

$= \frac{5}{3}\, u^{3/2}\Big]_1^2$

$= \frac{5}{3}[\sqrt{8} - 1]$

$= \frac{5}{3}[2\sqrt{2} - 1]$

$\frac{5}{2}\cdot\frac{2(1+t^2)^{3/2}}{3} = \frac{5}{3}[2^{3/2} - 1]$

b. $W = \int_C F\cdot dr \qquad F(r(t)) = \langle -\sin t, \cos t, \cos t\rangle$

$= \int_C F(r(t))\cdot r'(t)\,dt \qquad r'(t) = \langle -\sin t, 0, \cos t\rangle$

$= \int_0^{2\pi} \langle -\sin t, \cos t, \cos t\rangle \cdot \langle -\sin t, 0, \cos t\rangle\, dt$

$= \int_0^{2\pi} \sin^2 t + 0 + \cos^2 t\, dt$

$= \int_0^{2\pi} dt$

$= \boxed{2\pi}$

c. Conservative vector fields mean that any closed loop $\oint_C G\cdot dr = \boxed{0}$

conservative

$r(t) = \langle \cos t, 1, \sin t\rangle$ for $0 \leq t \leq 2\pi$

$r(0) = \langle 1, 1, 0\rangle$

$r(2\pi) = \langle 1, 1, 0\rangle$

$f(\text{start}) - f(\text{end}) = 0$

$f(\text{start}) = f(\text{end})$

if f is potential function s.t. $\nabla f = G$

$\int_C G\cdot dr = 0$

closed loop

4. Suppose a surface $\mathcal{S}$ is parametrized by $G(u,v) = (u+v^2, u^2, uv)$ for $-3 \le u \le 5$, $0 \le v \le 4$.

(a) (2 points) Calculate $\mathbf{T_u}$ and $\mathbf{T_v}$.

(b) (3 points) Set up the iterated integral in the u- and v-variables to calculate the surface area of $\mathcal{S}$. You do not need to evaluate the integral.

(c) (3 points) Find the tangent plane to $\mathcal{S}$ at $(2,1,1)$. $=P$

$u,v = 1,1$

$G(1,1) = (2,1,1)$ $N(1,1) = (2,1,-4)$

a. $\boxed{\begin{aligned} T_u &= \langle 1, 2u, v\rangle \\ T_v &= \langle 2v, 0, u\rangle \end{aligned}}$

$N = T_u \times T_v = \begin{vmatrix} i & j & k \\ 1 & 2u & v \\ 2v & 0 & u \end{vmatrix} = i[2u^2 - 0] - j[u - 2v^2] + k[-4uv]$

$= \langle 2u^2, -u+2v^2, -4uv\rangle$

$\|N\| = \sqrt{4u^4 + u^2 - 4uv^2 + 4v^4 + 16u^2v^2}$

b. $\iint_D \|N(u,v)\|\, du\,dv$ = surface area.

$= \int_{u=-3}^{5} \int_{v=0}^{4} \|\langle 2u^2, -u+2v^2, -4uv\rangle\|\, dv\,du$

$= \int_{v=0}^{4} \int_{u=-3}^{5} \sqrt{4u^4 + u^2 - 4uv^2 + 4v^4 + 16u^2v^2}\, du\,dv$

$G(u,v) = (2,1,1) = (u+v^2, u^2, uv)$

$uv = 1$

$-4uv = -4$ *

$2u^2 = 2$ *

$u^2 = 1$

$-1 + 2(2-1)$

$N = \langle 2(1), -3(1)+(4), -4(1)\rangle$

$= \langle 2, 1, -4\rangle$

c. $2u^2(x-2) + [-u+2v^2](y-1) - 4uv(z-1) = 0$

$2(x-2) + (y-1) - 4(z-1) = 0$

$2x - 4 + y - 1 - 4z + 4 = 0$

$\boxed{2x + y - 4z = 1}$

$u + v^2 = 2$

$v^2 = 2 - u$

$-u + 2v^2$

$= -u + 2[2-u]$

$= -u + 4 - 2u =$

$-3u + 4$

$= -3(1) + 4 = 1$

$uv = u^2$

$u(v = u)$

$u \ne -1$ since $v \ne 7$