Take-Home Final

Name: [redacted]

Student ID: [redacted]

Instructions:

- The take-home final will begin at 8 am March 17. You will be given **24 hours** to complete and submit your final. The submission window will be closed at 8 am March 18.
- **No late submission** will be considered. Make sure to allow enough time to complete and submit your works. **You must take the final exam in order to pass the class.** Make-ups for the final exam are permitted only under exceptional circumstances, as outlined in the UCLA student handbook.
- The exam will be open book/open notes. More specifically, you may **only use the textbook, lecture notes, and/or any materials uploaded to the CCLE course web page**. You must **show your work** to receive credit.
- You must **sign the code of conduct:**

> I assert, on my honor, that I have not received assistance of any kind from any other person while working on the final and that I have not used any non-permitted materials or technologies during the period of this evaluation.
>
> Signature: [redacted]

Any deviation from the rules may render your exam void. Also, if needed, you may be contacted after the exam and asked for additional explanations of solutions for problems on the final.

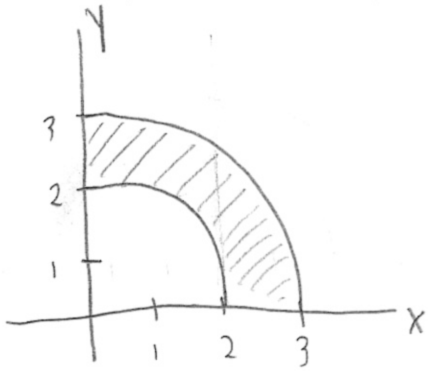
- **A Gradescope link for submitting your work will be provided.** Your submission should meet a set of criteria:
  (a) Your submission must be a single PDF file.
  (b) The code of conduct, your name, UID, and either physical or electronic signature must appear on the first page. (See above for an example.)
  (c) Starting from page 2, your solution to each of the problems must appear on a single page, in the order of the numbering. (For example, your solution to Question 1 must appear on page 2, Question 2 on page 3, and so forth.)

  There will be several ways to achieve this. The following is a set of common examples:
  - The exam template file will be designed to satisfy the above criteria. So you may simply print it out, fill in the necessary forms, and write down your solution to each of the problems. And then you may either scan it or take a (high-resolution and high-contrast) picture of it.
  - You may use letter size blank papers as soon as all the above criteria are met.
  - You may directly write on the exam PDF file, such as using a tablet.
  - You may use a word processor or LaTeX to prepare your submission electronically.

1. (10 pts) Compute

$$\int_0^2 \int_{\sqrt{4-x^2}}^{\sqrt{9-x^2}} e^{x^2+y^2}\, dydx + \int_2^3 \int_0^{\sqrt{9-x^2}} e^{x^2+y^2}\, dydx.$$

(Hint: Sketch the region on the xy-plane and convert the whole thing into one double integral.)



$$\int_0^2 \int_{\sqrt{4-x^2}}^{\sqrt{9-x^2}} e^{x^2+y^2}\, dydx + \int_2^3 \int_0^{\sqrt{9-x^2}} e^{x^2+y^2}\, dydx = \int_0^{\frac{\pi}{2}} \int_2^3 e^{r^2}\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} e^{r^2} \Big|_{r=2}^{3} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(e^9 - e^4)\, d\theta$$

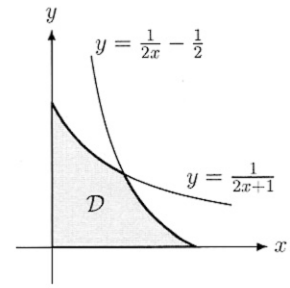
$$= \frac{\pi}{4}(e^9 - e^4)$$

2. (10 pts) Let $\mathcal{D}$ be the region in the first quadrant bounded by $x=0$, $y=0$, $y=\frac{1}{2x+1}$, and $y=\frac{1}{2x}-\frac{1}{2}$ as in the figure to the right. Use the change of variables

$$x=\frac{u}{1+v},\qquad y=\frac{v}{1+u},\qquad (\star)$$

to compute the integral

$$\iint_{\mathcal{D}}\frac{1}{1-xy}\,dxdy.$$



(Hint: The inverse map of $(\star)$ *is given by* $u=\frac{x(1+y)}{1-xy}$ *and* $v=\frac{y(1+x)}{1-xy}$*.)*

$$y=\frac{1}{2x}-\frac{1}{2}\rightarrow v=\frac{(\frac{1}{2x}-\frac{1}{2})(1+x)}{1-x(\frac{1}{2x}-\frac{1}{2})}=\frac{\frac{1}{2x}(1-x)(1+x)}{1-(\frac{1}{2}+\frac{x}{2})}=\frac{\frac{1}{x}(1-x)(1+x)}{1-x}=\frac{1+x}{x}$$

$$u=\frac{x(1+\frac{1}{2x}-\frac{1}{2})}{1-x(\frac{1}{2x}-\frac{1}{2})}=\frac{x+\frac{1}{2}-\frac{1}{2}x}{1-\frac{1}{2}+\frac{1}{2}x}=\frac{x+1}{1+x}=1$$

$$y=\frac{1}{2x+1}\rightarrow v=\frac{\frac{1}{2x+1}(1+x)}{1-x(\frac{1}{2x+1})}=\frac{1+x}{2x+1-x}=\frac{1+x}{1+x}=1$$

$$\mathrm{Jac}=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\begin{vmatrix}\frac{\partial x}{\partial u}&\frac{\partial x}{\partial v}\\\frac{\partial y}{\partial u}&\frac{\partial y}{\partial v}\end{vmatrix}$$

$$=\left(\frac{1}{1+v}\right)\left(\frac{1}{1+u}\right)-\left(\frac{-u}{(1+v)^2}\right)\left(\frac{-v}{(1+u)^2}\right)=\frac{(1+v)(1+u)-uv}{(1+v)^2(1+u)^2}=\frac{1+u+v}{(1+v)^2(1+u)^2}$$

$$y=\frac{1}{2x}-\frac{1}{2}\longrightarrow u=1\qquad y=\frac{1}{2x+1}\rightarrow v=1$$

$$y=0\longrightarrow v=0\qquad x=0\longrightarrow u=0$$

$$\iint_D\frac{1}{1-xy}\,dxdy=\int_0^1\int_0^1\frac{1}{1-(\frac{u}{1+v})(\frac{v}{1+u})}\left(\frac{1+u+v}{(1+v)^2(1+u)^2}\right)dudv$$

$$=\int_0^1\int_0^1\frac{1+u+v}{(1+v)^2(1+u)^2-uv(1+u)(1+v)}\,dudv$$

$$=\int_0^1\int_0^1\frac{1}{(1+u)(1+v)}\,\frac{1+u+v}{(1+u)(1+v)-uv}\,dudv$$

$$=\int_0^1\int_0^1\frac{1}{(1+u)(1+v)}\,\frac{1+u+v}{1+u+v}\,dudv$$

$$=\int_0^1\int_0^1\frac{1}{(1+u)(1+v)}\,dudv$$

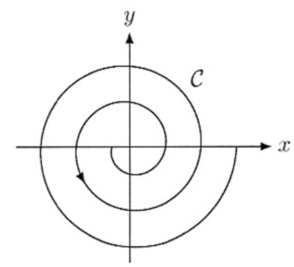
$$=\int_0^1\frac{1}{1+v}\ln(1+u)\Big|_{u=0}^{1}dv$$

$$=\int_0^1\frac{\ln(2)-\ln(1)}{1+v}\,dv$$

$$=\int_0^1\frac{\ln 2}{1+v}\,dv=(\ln 2)(\ln(1+v))\Big|_0^1=\ln 2[\ln(2)-\ln(1)]=[\ln(2)]^2$$

3. (10 pts) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x,y) = (\cos y)\mathbf{i} + (-x\sin y)\mathbf{j} + \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$



and C is the portion of the Archimedes' spiral, oriented counterclockwise, parametrized by

$$\mathbf{r}(t) = \langle t\cos t, t\sin t\rangle, \qquad \pi \le t \le 6\pi.$$

(Hint: Split **F** *into the sum of a conservative field and the vortex field.)*

Let $\vec{F}_1 = \langle \cos y, -x\sin y \rangle$ $\quad \vec{F}_2 = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$

$\frac{\partial f_1}{\partial x} = \cos y \qquad \frac{\partial f_1}{\partial y} = -x\sin y$

$f_1 = x\cos y + g(y) \qquad f_1 = x\cos y + h(x)$

$\Rightarrow f_1(x,y) = x\cos y + C$

$\vec{F}_2(\vec{r}(t)) = \langle \frac{-t\sin t}{t^2}, \frac{t\cos t}{t^2} \rangle$

$= \langle -\frac{\sin t}{t}, \frac{\cos t}{t} \rangle$

$\vec{r}\,'(t) = \langle \cos t - t\sin t, \sin t + t\cos t \rangle$

$\Rightarrow \int_C \vec{F}_1 \cdot d\vec{r} = f_1(\vec{r}(6\pi)) - f_1(\vec{r}(\pi)) = f_1(6\pi, 0) - f_1(-\pi, 0)$

$= 6\pi\cos(0) - (-\pi)\cos(0)$

$= 6\pi + \pi$

$= 7\pi$

$\vec{r}(6\pi) = \langle 6\pi, 0 \rangle$

$\vec{r}(\pi) = \langle -\pi, 0 \rangle$

$\int_C \vec{F}_2 \cdot d\vec{r} = \int_\pi^{6\pi} \langle -\frac{\sin t}{t}, \frac{\cos t}{t} \rangle \cdot \langle \cos t - t\sin t, \sin t + t\cos t \rangle \, dt$

$= \int_\pi^{6\pi} \frac{1}{t}(\sin t\cos t - \cos t\sin t) + \sin^2 t + \cos^2 t \, dt$

$= \int_\pi^{6\pi} 1 \, dt$

$= 6\pi - \pi = 5\pi$

$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}_1 \cdot d\vec{r} + \int_C \vec{F}_2 \cdot d\vec{r}$

$= 7\pi + 5\pi$

$= 12\pi$

4. (10 pts) Let $\mathcal{S}$ be the portion of the hyperbolic paraboloid $z = x^2 - y^2$ with $x^2 + y^2 \leq 4$, oriented with upward-pointing unit normal. Compute the flux of

$$\mathbf{F}(x, y, z) = \langle 2x + 3y, 3x - y, z \rangle$$

across $\mathcal{S}$.

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{s}$$

$$x^2 + y^2 \leq 4 \Rightarrow 0 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi$$

$$z = x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2\cos 2\theta$$

$$G(r, \theta) = (r\cos\theta, \ r\sin\theta, \ r^2\cos 2\theta)$$

$$T_r = \langle \cos\theta, \ \sin\theta, \ 2r\cos 2\theta \rangle$$

$$T_\theta = \langle -r\sin\theta, \ r\cos\theta, \ -2r^2\sin 2\theta \rangle$$

$$N = \langle -2r^2\sin\theta\sin 2\theta - 2r^2\cos\theta\cos 2\theta, \ -2r^2\sin\theta\cos 2\theta + 2r^2\cos\theta\sin 2\theta, \ r\cos^2\theta + r\sin^2\theta \rangle$$

$$= r^2\langle -2(\sin\theta\sin 2\theta + \cos\theta\cos 2\theta), -2(\sin\theta\cos 2\theta - \cos\theta\sin 2\theta), \tfrac{1}{r} \rangle$$

$$F(G(r,\theta)) = \langle 2r\cos\theta + 3r\sin\theta, \ 3r\cos\theta - r\sin\theta, \ r^2\cos 2\theta \rangle$$

$$\iint_S \vec{F} \cdot d\vec{s} = \int_0^{2\pi}\int_0^2 \Big(-2r^2(\sin\theta\sin 2\theta + \cos\theta\cos 2\theta)(2r\cos\theta + 3r\sin\theta) - 2r^2(\sin\theta\cos 2\theta - \cos\theta\sin 2\theta)(3r\cos\theta - r\sin\theta) + r^3\cos 2\theta\Big)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \Big[-2r^3(2\sin\theta\cos\theta\sin 2\theta + 3\sin\theta\cos\theta\cos 2\theta + 3\sin^2\theta\sin 2\theta + 2\cos^2\theta\cos 2\theta) - 2r^3(3\sin\theta\cos\theta\cos 2\theta$$
$$+ \sin\theta\cos\theta\sin 2\theta - 3\cos^2\theta\sin 2\theta - \sin^2\theta\cos 2\theta) + r^3\cos 2\theta\Big]\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 -6r^3\sin\theta\cos\theta\sin 2\theta - 12r^3\sin\theta\cos\theta\cos 2\theta - 6r^3\sin 2\theta(\sin^2\theta - \cos^2\theta) - r^3\cos 2\theta(4\cos^2\theta - 2\sin^2\theta) + r^3\cos 2\theta \, d\theta$$

$$= \int_0^2 r^3\,dr \int_0^{2\pi} -3\sin^2 2\theta - 6\sin 2\theta\cos 2\theta + 6\sin 2\theta\cos 2\theta - \cos 2\theta(2\cos^2\theta + 2\cos 2\theta - 1)\,d\theta$$

$$= \left[\tfrac{1}{4}r^4\right]_0^2 \int_0^{2\pi} -3\sin^2 2\theta - \cos 2\theta(1 + \cos 2\theta + 2\cos 2\theta - 1)\,d\theta$$

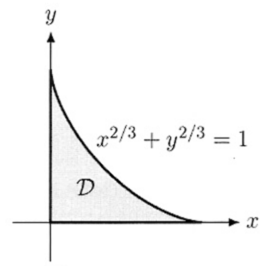
$$= 4\int_0^{2\pi} -3\sin^2 2\theta - 3\cos^2 2\theta \, d\theta$$

$$= 4\int_0^{2\pi} -3\,d\theta$$

$$= 4[-6\pi]$$

$$= -24\pi$$

5. (10 pts) Use the Green's Theorem to compute the area of the region $\mathcal{D}$ in the first quadrant bounded by $x = 0$, $y = 0$, and the curve $x^{2/3} + y^{2/3} = 1$ as in the figure to the right.



(Note: You might find $\int \cos^2 t \sin^2 t \, dt = \frac{1}{8}t - \frac{1}{32}\sin 4t + C$ helpful.)

Let $C_1 : y = 0, \; 0 \le x \le 1$ $\qquad \vec{r}_1(t) = \langle t, 0 \rangle \quad 0 \le t \le 1$

$C_2 : y = \sin^3 t, \; x = \cos^3 t$ $\qquad \vec{r}_2(t) = \langle \cos^3 t, \sin^3 t \rangle \quad 0 \le t \le \frac{\pi}{2}$

$C_3 : x = 0, \; 0 \le y \le 1$ $\qquad \vec{r}_3(t) = \langle 0, 1-t \rangle \quad 0 \le t \le 1$

$$\text{area}(D) = \frac{1}{2} \oint x\,dy - y\,dx$$

$$= \frac{1}{2}\left[\int_{C_1} x\,dy - y\,dx + \int_{C_2} x\,dy - y\,dx + \int_{C_3} x\,dy - y\,dx \right]$$

$$= \frac{1}{2}\left[\int_0^1 t(0) - 0(1)\,dt + \int_0^{\frac{\pi}{2}} \cos^3 t\,(3\cos t \sin^2 t) - \sin^3 t\,(-3\sin t \cos^2 t)\,dt + \int_0^1 0(-1) - (1-t)(0)\,dt \right]$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^2 t \sin^2 t\,(3\cos^2 t + 3\sin^2 t)\,dt$$

$$= \frac{3}{2} \int_0^{\frac{\pi}{2}} \cos^2 t \sin^2 t\,dt$$

$$= \frac{3}{2} \left[\frac{1}{8}t - \frac{1}{32}\sin 4t \right]_0^{\frac{\pi}{2}}$$

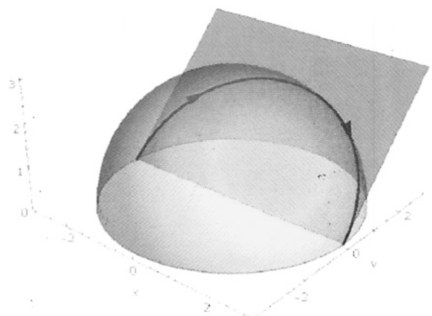
$$= \frac{3}{2} \left[\frac{\pi}{16} - \frac{1}{32}(0) \right]$$

$$= \frac{3\pi}{32}$$

6. (10 pts) Let C be the curve of intersection of the upper hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$ and the plane $z = y$, oriented clockwise when viewed from above. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = \langle x^2 + z, y + 2yz - x, y^2 + z^4 \rangle.$$

(Hint: Replace the path of integration to a simpler one using an appropriate theorem.)



Let S: area enclosed by C and the line $C_2: y = 0,\ -3 \leq x \leq 3$

$\vec{r}(t) = \langle x, 0, 0 \rangle$

$\vec{r}\,'(t) = \langle 1, 0, 0 \rangle$

$$\Rightarrow \iint_S \text{curl}(\vec{F}) \cdot dA = \int_C \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot dA - \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+z & y+2yz-x & y^2+z^4 \end{vmatrix} = \langle 2y - 2y, 1 - 0, -1 - 0 \rangle = \langle 0, 1, -1 \rangle$$

S: $0 \leq \theta \leq \pi$, $0 \leq r \leq 3$, $z = y$, downward facing normal

$G(r, \theta) = (r\cos\theta, r\sin\theta, r\sin\theta)$

$T_r = \langle \cos\theta, \sin\theta, \sin\theta \rangle$

$T_\theta = \langle -r\sin\theta, r\cos\theta, r\cos\theta \rangle$

$N = \langle r\sin\theta\cos\theta - r\sin\theta\cos\theta, -r\sin^2\theta - r\cos^2\theta, r\cos^2\theta + r\sin^2\theta \rangle = \langle 0, -r, r \rangle$

$\hookrightarrow N = \langle 0, r, -r \rangle$

$$\iint_S \text{curl}(\vec{F}) \cdot dA = \int_0^\pi \int_0^3 \langle 0, 1, -1 \rangle \cdot \langle 0, r, -r \rangle \, dr\, d\theta$$

$$= \int_0^\pi \int_0^3 2r \, dr\, d\theta = \pi \left[r^2 \right]_0^3 = 9\pi$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_3^{-3} \langle x^2, -x, 0 \rangle \cdot \langle 1, 0, 0 \rangle \, dx = \int_3^{-3} x^2 \, dx = \left[\tfrac{-1}{3} x^3 \right]_{-3}^{3} = -9 + (-9) = -18$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot dA - \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= 9\pi + 18$$

7. (10 pts) Let $\mathcal{S}$ be that part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the two planes $z = 2$ and $z = 4$ with upward-pointing unit normal. Suppose that the vector field $\mathbf{F}$ has the vector potential $\mathbf{A}$ given by

$$\mathbf{A}(x, y, z) = \langle -yz^2, xz^2, xyz \rangle.$$ ↗ clockwise

Compute the flux $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

C_1: circle radius 2, $z=2$

C_2: circle radius 4, $z=4$

↘ counterclockwise

$$\iint_S \vec{F} \cdot ds = \oint_{\partial S} \vec{A} \cdot d\vec{r}$$

$\vec{r}_1(t) = <2\cos t, 2\sin t, 2>$

$\vec{r}_1{}'(t) = <-2\sin t, 2\cos t, 0>$

$\vec{A}(\vec{r}_1(t)) = <-8\sin t, 8\cos t, 8\cos t \sin t>$

$\vec{r}_2(t) = <4\cos t, 4\sin t, 4>$

$\vec{r}_2{}'(t) = <-4\sin t, 4\cos t, 0>$

$\vec{A}(\vec{r}_2(t)) = <-64\sin t, 64\cos t, 64\cos t \sin t>$

$$\int_{C_1} \vec{A} \cdot d\vec{r} = \int_{2\pi}^{0} <-8\sin t, 8\cos t, 8\cos t \sin t> \cdot <-2\sin t, 2\cos t, 0> \cdot dt$$

$$= \int_{2\pi}^{0} 16\sin^2 t + 16\cos^2 t \, dt$$

$$= -32\pi$$

$$\int_{C_2} A \cdot d\vec{r} = \int_0^{2\pi} <-64\sin t, 64\cos t, 64\cos t \sin t> \cdot <-4\sin t, 4\cos t, 0> dt$$

$$= \int_0^{2\pi} 256\sin^2 t + 256\cos^2 t \, dt$$

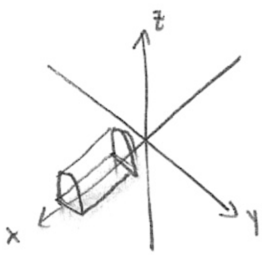
$$= 512\pi$$

$$\oint_{\partial S} \vec{A} \cdot d\vec{r} = \int_{C_1} \vec{A} \cdot d\vec{r} + \int_{C_2} \vec{A} \cdot d\vec{r} = -32\pi + 512\pi = 480\pi$$

$$\iint_S \vec{F} \cdot ds = 480\pi$$

8. (10 pts) Let $\mathcal{S}$ be the boundary of that part of the solid elliptical cylinder $y^2 + 4z^2 \leq 16$ with $1 \leq x \leq 3$ and $z \geq 0$. Use the Divergence Theorem to compute the flux $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = \langle 3xy, 3yz, 3zx \rangle.$$



$y^2 + 4z^2 = 16$

$\left(\frac{y}{4}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$

$y = 4\cos\theta, \quad z = 2\sin\theta \quad 0 \leq \theta \leq \pi$

$G(r, \theta, x) = (x, 2r\cos\theta, r\sin\theta) \quad 1 \leq x \leq 3, \ 0 \leq r \leq 2, \ 0 \leq \theta \leq \pi$

$\text{div}(\vec{F}) = 3y + 3z + 3x$

$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial x} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial x} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial r} & \frac{\partial z}{\partial x} \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 2\cos\theta & -2r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \end{vmatrix}$

$= 2r\cos^2\theta + 2r\sin^2\theta = 2r$

$\iint_S F \cdot ds = \iiint_W \text{div}(\vec{F})\, dV \quad \text{where } S = \partial W$

$= \int_0^{\pi} \int_0^2 \int_1^3 \left(3(2r\cos\theta) + 3(r\sin\theta) + 3x\right) 2r \, dx\, dr\, d\theta$

$= \int_0^{\pi} \int_0^2 \int_1^3 12r^2\cos\theta + 6r^2\sin\theta + 6rx \, dx\, dr\, d\theta$

$= \int_0^{\pi} \int_0^2 24r^2\cos\theta + 12r^2\sin\theta + 3r(8)\, dr\, d\theta$

$= \int_0^{\pi} \left[8r^3\cos\theta + 4r^3\sin\theta + 12r^2\right]_{r=0}^{2} d\theta$

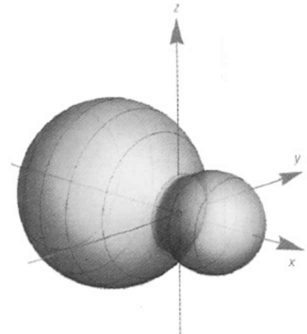
$= \int_0^{\pi} 64\cos\theta + 32\sin\theta + 48 \, d\theta$

$= \left[64\sin\theta - 32\cos\theta + 48\theta\right]_0^{\pi}$

$= 32 + 48\pi - (-32)$

Flux $= 64 + 48\pi$

9. (10 pts) Let $\mathcal{S}$ be the snowman-like closed surface as in the figure to the right, which is oriented with outward-pointing unit normal. Suppose you know that the unit ball $\mathcal{B} = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$ is entirely contained in the interior of $\mathcal{S}$. Use the Divergence Theorem to compute the flux $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}$ is given by



$$\mathbf{F}(x, y, z) = \frac{3\langle x, y, z\rangle}{\rho^3}, \qquad \rho = \sqrt{x^2 + y^2 + z^2}.$$

$$\operatorname{div}(\vec{F}) = \frac{3}{\rho^3} - \frac{9x}{\rho^4}\frac{\partial \rho}{\partial x} + \frac{3}{\rho^3} - \frac{9y}{\rho^4}\frac{\partial \rho}{\partial y} + \frac{3}{\rho^3} - \frac{9z}{\rho^4}\frac{\partial \rho}{\partial z}$$

$$= \frac{9}{\rho^3} - \frac{9}{\rho^4}\left(x\frac{\partial \rho}{\partial x} + y\frac{\partial \rho}{\partial y} + z\frac{\partial \rho}{\partial z}\right)$$

$$= \frac{9}{\rho^3} - \frac{9}{\rho^4}\left(\frac{x^2}{\sqrt{x^2+y^2+z^2}} + \frac{y^2}{\sqrt{x^2+y^2+z^2}} + \frac{z^2}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{9}{\rho^3} - \frac{9}{\rho^4}\frac{\rho^2}{\rho} = 0$$

* $\operatorname{div}(\vec{F}) = 0$ at all points except the origin because $\vec{F}$ is not defined at the origin

Let W_1 be the volume enclosed between S and the unit ball
W_2 be the volume enclosed by the unit ball

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial W_1} \vec{F} \cdot d\vec{S} + \iint_{\partial W_2} \vec{F} \cdot d\vec{S}$$

∂W_2: $G(\theta, \phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$
$T_\theta = \langle -\sin\theta\sin\phi, \cos\theta\sin\theta, 0\rangle$
$T_\phi = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle$
$N = \langle -\cos\theta\sin^2\phi, -\sin\theta\sin^2\phi, -\sin\phi\cos\phi\rangle$

$$= \iiint_{W_1} \operatorname{div}(\vec{F})\,dV + \int_0^{2\pi}\int_0^{\pi} 3\langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle \cdot \langle \cos\theta\sin^2\phi, \sin\theta\sin^2\phi, \sin\phi\cos\phi\rangle\, d\phi\, d\theta$$

$$= \iiint_{W_1} 0\,dV + \int_0^{2\pi}\int_0^{\pi} 3\cos^2\theta\sin^3\phi + 3\sin^2\theta\sin^3\phi + 3\sin\phi\cos^2\phi\, d\phi\, d\theta$$

$$= 0 + 3\int_0^{2\pi}\int_0^{\pi} \sin\phi(\sin^2\phi + \cos^2\phi)\, d\phi\, d\theta$$

$$= 3\int_0^{2\pi}\int_0^{\pi} \sin\phi\, d\phi\, d\theta$$

$$= 3\,(2\pi)\left[-\cos\phi\right]_0^{\pi}$$

$$\iint_S \vec{F} \cdot d\vec{S} = 12\pi$$