Math 32B | Midterm 2 | Spring 2013

Calculus of Several Variables

May 20, 2013

Answer the questions in the spaces provided. If you run out of room use scratch paper and attach it to end of the exam. Show your work. Correct answers not accompanied by sufficient explanations will receive little or no credit. Please ask any of the proctors if you have any questions about a problem. Calculators are allowed but definitely not needed. No books, PDAs, cell phones or other devices (other than calculators) will be permitted.

Question	Points	Score
1	25	16
2	20	13
3	30	30
4	25	25
Total:	100	~~[illegible]~~ 84

Student Name: ______________________________

Student Signature: ______________________________

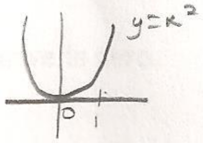
Section: ____ Meets on: ______ TA Name: _______________

(15 pts) 1. A) The *Flux* of a vector field $\vec{F}(x,y) = P(x,y)\hat{\imath} + Q(x,y)\hat{\jmath}$ *across an oriented curve* C is given by

$$\text{Flux}_C(\vec{F}) = \int_C P(x,y)\,\mathrm{d}y - Q(x,y)\,\mathrm{d}x.$$

Calculate the flux of $\vec{F}(x,y) = \mathrm{e}^y\hat{\imath} + (2x-1)\hat{\jmath}$ across the parabola $y = x^2$, $0 \le x \le 1$, oriented from left to right.

$F = \langle e^y, 2x-1 \rangle$



$\cancel{c(t) = \langle t, t^2 \rangle \; 0 \le t \le 1}$

$\cancel{c'(t) = \langle 1, 2t \rangle}$

$c(x) = \langle x, x^2 \rangle \quad 0 \le x \le 1$

$c'(x) = \langle 1, 2x \rangle$

+8

$Flux = \int_0^1 e^{2x}(2x) - (2-1)(1) = \int_0^1 2xe^x - 1\,dx$

$u = 2x \qquad v = e^x$

$du = 2 \qquad dv = e^x$

$= 2xe^x - \int_0^1 2e^x + 1\,dx = 2xe^x - 2e^x - x \Big|_0^1 = 2e - 2e - 1 - 0 + 2 + 0 = 1$

+2

(10 pts) B) I. Which of the following statements are true for all vector fields, and which are true only for conservative vector fields?

i) The line integral along a path from P to Q does not depend on which path is chosen.

i) true for Conservative

ii) The line integral around a closed curve is zero.

ii) true for Conservative

iii) The cross-partials of the components are equal.

iii) true for Conservative

iv) The line integral over an oriented curve C does not depend on how C is parametrized.

iv) true for all

v) The line integral is equal to the difference of a potential function at the two endpoints.

v) true for Conservative

vi) The line integral is equal to the integral of the tangential component along the curve.

vi) ~~true~~ for Conservative

vii) The line integral changes signs if the orientation is reversed.

vii) true for all

II. Let $\vec{F}$ be a vector field on an open, connected domain $\mathcal{D}$. Which of the following statements are always true, and which are true under additional hypothesis on $\mathcal{D}$?

i) If $\vec{F}$ has a potential function, then $\vec{F}$ is conservative.

i) true

ii) If $\vec{F}$ is conservative, then the cross-partials of $\vec{F}$ are equal.

ii) true

iii) If the cross-partials of $\vec{F}$ are equal, then $\vec{F}$ is conservative.

iii) true only if D is also simply-connected

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(20 pts) 2. Calculate the integral of $f(x, y, z) = e^z$ over the portion of the plane $x + 2y + 2z = 3$, where $x, y, z \geq 0$.

HINT:

i) An appropriate choice of the order of integration might help you avoid having to integrate by parts.

The answer is $3(e^{\frac{3}{2}} - \frac{5}{2})$.

for $z \geq 0$

$0 \leq x \leq 3$

$0 \leq y \leq 3/2$

$G(x,y) = (x, y, 3/2 - x/2 - y)$

$T_x = (1, 0, -1/2)$

$T_y = (0, 1, -1)$

$N = T_x \times T_y = (1/2, 1, 1)$

$\|N\| = \sqrt{1/4 + 1 + 1} = \sqrt{9/4} = 3/2$

$$\int_0^3 \int_0^{3/2} \frac{3}{2} e^{3/2 - x/2 - y} \, dy\,dx = \int_0^3 \left(-\frac{3}{2} e^{3/2} e^{-x/2} e^{-y} \Big|_0^{3/2} \right) dx = \int_0^3 -\frac{3}{2} e^{-x/2} \, dx$$

$$= 3e^{-x/2} \Big|_0^3 = 3(e^{-3/2} + 1)$$

$0 \leq y \leq \frac{3}{2} - \frac{x}{2}$

$$\int_0^3 -\frac{3}{2} e^{3/2 - x/2 - 3/2 + x/2} + \frac{3}{2} e^{3/2 - x/2} \, dx$$

$$= \int_0^3 -\frac{3}{2} - \frac{3}{2} e^{3/2 - x/2} \, dx = -\frac{3}{2} \int_0^3 1 + e^{3/2 - x/2} \, dx$$

$$= -\frac{3}{2} \left(x - 2e^{3/2 - x/2} \Big|_0^3 \right) = -\frac{3}{2}(3 - 2 - 0 + 2e^{3/2})$$

$$= 3(-e^{3/2} - \frac{1}{2})$$

(30 pts) 3. The *Flux* of a vector field $\vec{F}(x,y,z)$ *across a closed surface* S is given by the surface integral of $\vec{F}$ over S. In symbols

$$\mathrm{Flux}_S(\vec{F}) = \iint_S \vec{F}\cdot \mathrm{d}\vec{S},$$

where the normal unit vector $\hat{\boldsymbol{n}}$ that determines the orientation of S is the one that points towards the outside of the volume enclosed by S.

Let $\vec{F} = y\hat{\imath} + yz\hat{\jmath} + (z^2 - 5z)\hat{\boldsymbol{k}}$, compute $\mathrm{Flux}_S(\vec{F})$ where S is the boundary of the cylinder shown in Figure 1. Keep in mind that S also includes the top and the bottom of the cylinder.

HINTS:

i) Figuring out the normal component of the vector field on the top and the bottom of the cylinder might help you avoid unnecessary computations.

ii) The following integrals might come in handy:
$\int_0^{2\pi} \sin t \cos t\,\mathrm{d}t = \int_0^{2\pi} \sin mt\,\mathrm{d}t = \int_0^{2\pi} \cos mt\,\mathrm{d}t = 0$, for $m = 1, 2, \ldots$
$\int_0^{2\pi} \sin^2 t\,\mathrm{d}t = \int_0^{2\pi} \cos^2 t\,\mathrm{d}t = \pi$.

$F = \langle y, yz, z^2-5z \rangle$ $\qquad x^2+y^2=4$

$G(\theta, z) = (2\cos\theta, 2\sin\theta, z)$

$T_\theta = (-2\sin\theta, 2\cos\theta, 0)$

$T_z = (0, 0, 1)$

$N = T_\theta \times T_z = (2\cos\theta, 2\sin\theta, 0)$

- because N has no $\hat{k}$ component, the flux across the top and bottom of the cylinder is equal to zero, so total flux only depends on the sides of the cylinder

$\mathrm{Flux} = \int_0^{2\pi}\int_0^5 F(G)\cdot N\,dz\,d\theta = \int_0^{2\pi}\int_0^5 \langle 2\sin\theta, 2z\sin\theta, z^2-5z\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle\,dz\,d\theta$

$= \int_0^{2\pi}\int_0^5 4\sin\theta\cos\theta + 4z\sin^2\theta\,dz\,d\theta = \int_0^{2\pi} 20\sin\theta\cos\theta + 50\sin^2\theta\,d\theta$

$= 20(0) + 50(\pi) = 50\pi$

30

(25 pts) 4. Let C be a curve in polar form $r = f(\theta)$ for $\theta_1 \le \theta \le \theta_2$ (see Figure 2), parametrized by

$$\vec{c}(\theta) = (r\cos\theta, r\sin\theta) = \big(f(\theta)\cos\theta, f(\theta)\sin\theta\big).$$

i) Show that $\vec{c}\,'(\theta) = f'(\theta)\hat{e}_r + f(\theta)\hat{e}_\theta$, where $\hat{e}_r = (\cos\theta, \sin\theta)$ and $\hat{e}_\theta = (-\sin\theta, \cos\theta)$.

ii) Show that if $L(C) = \int_C 1\,\mathrm{d}s$, represents the length of C, then

$$L(C) = \int_{\theta_1}^{\theta_2} \sqrt{\big(f(\theta)\big)^2 + \big(f'(\theta)\big)^2}\,\mathrm{d}\theta$$

HINTS: $\hat{e}_r \perp \hat{e}_\theta$, $||\hat{e}_r|| = ||\hat{e}_\theta|| = 1$, and $||\vec{c}\,'(\theta)||^2 = \vec{c}\,'(\theta)\cdot\vec{c}\,'(\theta)$.

iii) Use polar coordinates to show that along C, the vortex vector field

$$\vec{F}(x,y) = \frac{-y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$$

is given by $\vec{F}\big(\vec{c}(\theta)\big) = \frac{1}{f(\theta)}\hat{e}_\theta$ and conclude that

$$\int_C \vec{F}\cdot\mathrm{d}\vec{s} = \theta_2 - \theta_1.$$

i) $c(\theta) = \langle f(\theta)\cos\theta, f(\theta)\sin\theta\rangle$

$c'(\theta) = \langle f(\theta)(-\sin\theta) + \cos\theta f'(\theta), f(\theta)\cos\theta + \sin\theta f'(\theta)\rangle$

$= \langle f'(\theta)\cos\theta - f(\theta)\sin\theta, f'(\theta)\sin\theta + f(\theta)\cos\theta\rangle$ 5

$= f'(\theta)\langle\cos\theta, \sin\theta\rangle + f(\theta)\langle -\sin\theta, \cos\theta\rangle = f'(\theta)\hat{e}_r + f(\theta)\hat{e}_\theta$

ii) $\|c'(\theta)\| = \sqrt{f'(\theta)^2\cos^2\theta - 2f'(\theta)f(\theta)\sin\theta\cos\theta + f(\theta)^2\sin^2\theta + f'(\theta)^2\sin^2\theta + 2f'(\theta)f(\theta)\sin\theta\cos\theta + f(\theta)^2\cos^2\theta}$

$= \sqrt{f'(\theta)^2\cos^2\theta + f'(\theta)^2\sin^2\theta + f(\theta)^2\sin^2\theta + f(\theta)^2\cos^2\theta}$

$= \sqrt{(f'(\theta))^2 + (f(\theta))^2}$ 10

Since arc length $= \int_C f(x)\|f'(x)\|\,dx$

$\int_C 1\,ds = \int_{\theta_1}^{\theta_2} 1\sqrt{(f'(\theta))^2 + (f(\theta))^2}\,d\theta = \int_{\theta_1}^{\theta_2}\sqrt{(f(\theta))^2 + (f'(\theta))^2}\,d\theta$

iii) on next page →

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iii) $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ $\quad c(\theta) = \langle f(\theta)\cos\theta, f(\theta)\sin\theta \rangle$

$$\vec{F}(c(\theta)) = \left\langle \frac{-f(\theta)\sin\theta}{f(\theta)^2\cos^2\theta + f(\theta)^2\sin^2\theta}, \frac{f(\theta)\cos\theta}{f(\theta)^2\cos^2\theta + f(\theta)^2\sin^2\theta} \right\rangle$$

$$= \left\langle \frac{-f(\theta)\sin\theta}{f(\theta)^2}, \frac{f(\theta)\cos\theta}{f(\theta)^2} \right\rangle$$

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$$= \left\langle \frac{-\sin\theta}{f(\theta)}, \frac{\cos\theta}{f(\theta)} \right\rangle = \frac{1}{f(\theta)} \langle -\sin\theta, \cos\theta \rangle$$

$$= \frac{1}{f(\theta)} \hat{e}_\theta \quad \text{where } \hat{e}_\theta = \langle -\sin\theta, \cos\theta \rangle$$

$$\int_c F \cdot ds = \int_c F(c(\theta)) \cdot c'(\theta) = \int_{\theta_1}^{\theta_2} \frac{1}{f(\theta)} \hat{e}_\theta \cdot \left(f'(\theta)\hat{e}_r + f(\theta)\hat{e}_\theta \right) d\theta$$

$$= \int_{\theta_1}^{\theta_2} \frac{f'(\theta)}{f(\theta)} \hat{e}_\theta \cdot \hat{e}_r + \frac{f(\theta)}{f(\theta)} \hat{e}_\theta \cdot \hat{e}_\theta \, d\theta$$

5

- since $\hat{e}_\theta \perp \hat{e}_r$, $\hat{e}_\theta \cdot \hat{e}_r = 0$
- $\hat{e}_\theta \cdot \hat{e}_\theta = \|\hat{e}_\theta\|^2$, so since $\|\hat{e}_\theta\| = 1$, $\hat{e}_\theta \cdot \hat{e}_\theta = 1$

$$= \int_{\theta_1}^{\theta_2} \frac{f'(\theta)}{f(\theta)}(0) + \frac{f(\theta)}{f(\theta)}(1) \, d\theta = \int_{\theta_1}^{\theta_2} 1 \, d\theta = \theta_2 - \theta_1$$

very good explanation! :)