MATH 32B-1, Fall 2016, MIDTERM II
Calculus of Several Variables

Name: ....

UID number:..................................

Section & TA: ....SOLUTIONS...

DO NOT START UNTIL TOLD TO DO SO

You have 50 minutes to complete the exam. There are 4 problems, worth a total of 100 points.

No books, calculators, or notes of any kind are allowed.

Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

1	
2	
3	
4	
Total	

Exercise 1 (25 points)

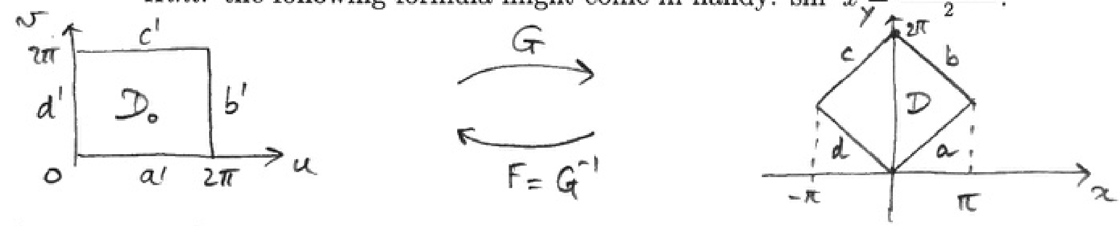
(a) [10 pts.] Let $\mathcal{D}$ be the diamond-shaped region in $\mathbb{R}^2$ with vertices at $(0,0)$, (π,π), $(0,2\pi)$, $(-\pi,\pi)$. Find a map which transforms the region $\mathcal{D}_0 = [0,2\pi]\times[0,2\pi]$ in the (u,v)-plane into $\mathcal{D}$. Make a picture of both $\mathcal{D}_0$ and $\mathcal{D}$.

(b) [15 pts.] Use your answer from part (a) to evaluate the following integral:

$$\int\int_{\mathcal{D}} (x-y)^2 \sin^2(x+y)\, dA.$$

Hint: the following formula might come in handy: $\sin^2 x = \frac{1-\cos(2x)}{2}$.

(a)



a has equation $y = x$ for $0 \le x \le \pi$

c has equation $y = 2\pi + x$ for $-\pi \le x \le 0$

b has equation $y = 2\pi - x$ for $0 \le x \le \pi$

d has equation $y = -x$ for $-\pi \le x \le 0$

therefore, let us define:

$$F: \begin{cases} u = x + y \\ v = y - x \end{cases} \Rightarrow \begin{array}{ll} a \to a' : & v = 0 \\ c \to c' : & v = 2\pi \\ b \to b' : & u = 2\pi \\ d \to d' : & u = 0 \end{array}$$

(b) Let us compute the Jacobian of $F = G^{-1}$

$$\mathrm{Jac}(F) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 2$$

$$\Rightarrow \quad \mathrm{Jac}(G) = \left(\mathrm{Jac}(G^{-1})\right)^{-1} = \left[\mathrm{Jac}(F)\right]^{-1} = 1/2$$

$$\Rightarrow \iint_{\mathcal{D}} (x-y)^2 \sin^2(x+y)\, dA = \iint_{\mathcal{D}_0} (-v)^2 \sin^2 u \cdot \frac{1}{2} \cdot du\, dv = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} v^2 \sin^2 u \, du\, dv =$$

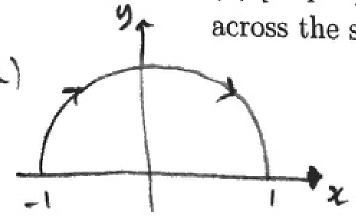
$$= \frac{1}{2}\left(\int_0^{2\pi} v^2\, dv\right)\cdot \underbrace{\left(\int_0^{2\pi} \sin^2 u\, du\right)}_{\text{use the formula for } \sin^2 u} = \frac{1}{2} \cdot \left.\frac{v^3}{3}\right|_0^{2\pi} \cdot \left(\frac{1}{2}\cdot 2\pi - \left.\frac{\sin 2x}{4}\right|_{x=0}^{2\pi}\right) =$$

$$= \frac{1}{2}\cdot\frac{2^3\pi^3}{3}\cdot\pi = \boxed{\frac{4}{3}\pi^4}$$

Exercise 2 (24 points)

(a) [12 pts.] Find the total charge on the upper semicircle $x^2+y^2=1$, $y \geq 0$, oriented clockwise, with charge density $\delta(x,y)=xy^3$;
(b) [12 pts.] Find the flux of the vector field $\mathbf{F} = \left(\frac{y^3}{[(x+2)^4+y^4]^{1/2}}, \frac{(x+2)^3}{[(x+2)^4+y^4]^{1/2}}\right)$ across the segment $1 \leq x \leq 3$ oriented left to rigth.

(a)



let C be parametrized by ① $\underline{r}(t) = (-\cos t, \sin t)$ $0 \leq t \leq \pi$

(But you could also use, for example, $\underline{r}(t) = (\sin t, \cos t)$ $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$)

let us adopt ① $\Rightarrow$ $\underline{r}'(t) = (\sin t, \cos t)$ $\Rightarrow$ $\|\underline{r}'(t)\| = 1$

$$\text{Total charge} = \int_C \delta(x,y,z)\, ds = \int_{\underline{r}(t)} \delta(\underline{r}(t)) \cdot \|\underline{r}'(t)\| \cdot dt = \int_0^{\pi} (-\cos t)(\sin t)^3\, dt =$$

$$= -\int_0^{\pi} (\sin t)^3 \frac{d}{dt}(\sin t)\, dt = -\frac{1}{4}\int_0^{\pi} \frac{d}{dt}(\sin t)^4\, dt = -\frac{1}{4}(\sin t)^4 \Big|_0^{\pi} = \boxed{0}$$

In fact xy^3 is an odd function of x over a domain which is x-symmetric.

Moreover $ds = \|\underline{r}'(t)\|$ does not change sign with x $\Rightarrow$ We expect the integral to be zero!

(b)

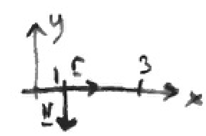
$$\text{Flux of } \underline{F} = \int_C \underline{F} \cdot \underline{n}\, ds = \int_C \underline{F} \cdot \underline{N}\, dt$$

the line segment $1 \leq x \leq 3$ can be parametrized as follows:

$\underline{r}(t) = (t, 0)$ $1 \leq t \leq 3$

$\underline{r}'(t) = (1, 0)$

$\underline{N}(t) = (0, -1)$



$\Rightarrow$

$$\text{Flux of } \underline{F} = \int_1^3 \underline{F}(\underline{r}(t)) \cdot \underline{N}(t)\, dt =$$

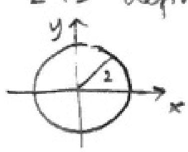
$$= \int_1^3 \left(0, \frac{(t+2)^3}{[(t+2)^4+0]^{1/2}}\right) \cdot (0,-1)\, dt = -\int_1^3 (t+2)\, dt = \left[\frac{t^2}{2} + 2t\right]_{t=3}^{t=1} =$$

$$= \frac{1}{2} - \frac{9}{2} + 2 - 6 = \boxed{-8}$$

Exercise 3 (26 points)

(a) [10 pts.] Let $\mathbf{F}(x,y) = (\frac{-y+x}{x^2+y^2}, \frac{x+y}{x^2+y^2})$ be a planar vector field. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the circle $x^2+y^2=4$ oriented counterclockwise.
(b) [8 pts.] Is $\mathbf{F}$ conservative on $\mathcal{D} = \{(x,y) \neq (0,0)\}$? Explain.
(c) [4 pts.] Show that $\mathbf{F}$ satisfies the cross-partial condition.
(d) [4 pts.] Show that $\mathbf{F}$ is conservative on $\mathcal{D} = \{(x,y) | x > 0\}$

(a) $\underline{F}$ is defined on $\mathbb{R}^2 - \{(0,0)\}$.



$\underline{r}(t) = (2\cos t, 2\sin t) \qquad 0 \le t \le 2\pi$

$\underline{r}'(t) = (-2\sin t, 2\cos t)$

$$\oint \underline{F}\cdot d\underline{r} = \frac{1}{4}\int_0^{2\pi} \left(2(\cos t - \sin t), 2(\cos t + \sin t)\right)\cdot(-2\sin t, 2\cos t)\, dt =$$

$$= \int_0^{2\pi} \left(-\cancel{\sin t\cos t} + \sin^2 t + \cos^2 t + \cancel{\sin t \cos t}\right) dt = \int_0^{2\pi} dt = 2\pi \neq 0$$

(b) No, $\underline{F}$ cannot be conservative, since we found one closed path around which the circulation of $\underline{F}$ is different from zero. (and we know that $\underline{F}$ is conservative $\Leftrightarrow \oint \underline{F}\cdot d\underline{r} = 0$ around ANY closed path).

(c)
$$\frac{\partial F_1}{\partial y} = \frac{-(x^2+y^2)-(x-y)2y}{(x^2+y^2)^2} = \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial F_2}{\partial x} = \frac{x^2+y^2-(x+y)(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2-2xy}{(x^2+y^2)^2}$$

the domain $\mathbb{R}^2 - \{(0,0)\}$ is not simply connected, so the condition $\underline{\nabla}\times\underline{F} = \underline{0}$ is not sufficient to guarantee that $\underline{F}$ is conservative. (and, in fact, it is NOT).

(d) Since the domain $D = \{(x,y) \mid x > 0\}$ is simply connected and we already know that $\underline{\nabla}\times\underline{F} = \underline{0}$, $\underline{F}$ is conservative on $D = \{(x,y) \mid x>0\}$.
Let us find the potential function $f(x,y)$ such that $\underline{F} = \underline{\nabla} f$. It has to be:

$$\frac{\partial f}{\partial x} = F_1 = \frac{x-y}{x^2+y^2} \Rightarrow f(x,y) = \int \frac{x}{x^2+y^2}\,dx - \int \frac{y}{x^2+y^2}\,dx + c(y) =$$

$$= \frac{1}{2}\ln(x^2+y^2) - \int \frac{y/x^2}{1+(y/x)^2}\,dx + c(y) =$$

$$= \frac{1}{2}\ln(x^2+y^2) + \tan^{-1}(y/x) + c(y).$$

Moreover:

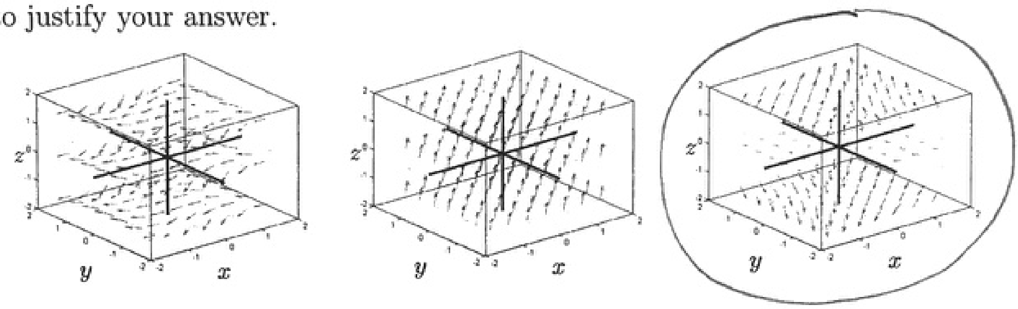
$$\frac{\partial f}{\partial y} = F_2 = \frac{x+y}{x^2+y^2} \qquad \frac{\partial f}{\partial y} = \frac{y}{x^2+y^2} + \frac{1/x}{1+(y/x)^2} = \frac{y+x}{x^2+y^2} + \frac{dc}{dy} \Rightarrow \frac{dc}{dy} = 0$$

therefore $\quad c(y) \equiv k$

$$\Rightarrow \boxed{f(x,y) = \frac{1}{2} \ln\left(x^2+y^2\right) + \tan^{-1}\left(\frac{y}{x}\right) + k}$$

Exercise 4 (25 points)

(a) [5 pts.] Given the three-dimensional vector field $\mathbf{F}(x,y,z) = (\frac{y}{1+x^2}, \tan^{-1} x, 2z)$ which of the following is a plot of $\mathbf{F}$? Circle the right one, you do not need to justify your answer.



(b) [5 pts.] Compute div($\mathbf{F}$) and curl($\mathbf{F}$);

(c) [15 pts.] Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ over the unit circle in the (x,y)-plane clockwise oriented.

(b) $\operatorname{div} \underline{F} = \nabla \cdot \underline{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 = -\frac{2xy}{(1+x^2)^2} + 0 + 2 = 2 - \frac{2xy}{(1+x^2)^2}$

$$\operatorname{curl} \underline{F} = \nabla \times \underline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{1+x^2} & \tan^{-1} x & 2z \end{vmatrix} = \underline{i} \cdot 0 + \underline{j} \cdot 0 + \underline{k} \underbrace{\left(\frac{\partial}{\partial x} \tan^{-1} x - \frac{\partial}{\partial y} \frac{y}{1+x^2} \right)}_{= 0} = \boxed{0}$$

Since $\underline{F}$ is defined on $\mathbb{R}^3$, which is a simply connected domain, and it is curl-free $\Rightarrow$ $\underline{F}$ is conservative.

Let us find the potential function:

$\frac{\partial f}{\partial x} = F_1 = \frac{y}{1+x^2} \Rightarrow f(x,y,z) = \int \frac{y}{1+x^2} dx + c_1(y,z) = y \tan^{-1} x + c_1(y,z)$

but also

$\frac{\partial f}{\partial y} = F_2 = \tan^{-1} x \Rightarrow \frac{\partial f}{\partial y} = \tan^{-1}(x) + \frac{\partial c_1}{\partial y}(y,z) \Rightarrow \frac{\partial c_1}{\partial y} = 0 \Rightarrow$ c_1 does NOT depend on $y \Rightarrow$ it is a function of z only!

$c_1(y,z) \equiv c_1(z)$

$\Rightarrow f(x,y,z) = y \tan^{-1} x + c_1(z)$

Since $\frac{\partial f}{\partial z} = F_3 = 2z$ and $\frac{\partial f}{\partial z} = \frac{dc_1}{dz} \Rightarrow \frac{dc_1}{dz} = 2z \Rightarrow c_1(z) = \int 2z \, dz + k = z^2 + k$

$\Rightarrow \boxed{f(x,y,z) = y \tan^{-1} x + z^2 + k}$ 5

(c) the integral of a conservative field along ANY closed path is zero $\Rightarrow \boxed{\oint \underline{F} \cdot d\underline{r} = 0}$