1	4
2	4
3	4
4	4
5	4
T	20

MATH 32B Midterm II, Winter 2012

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Problem 1. (4)

Evaluate the following line integral

$$\int_C 2y\,dx - x\,dy$$

by two different methods:(a) compute it directly; (b) use the Green theorem to compute it. Here C is the unit circle $x^2 + y^2 = 1$ with the counter clockwise orientation.

a) $x = \cos\theta \quad y = \sin\theta \qquad 0 \le \theta \le 2\pi$

$\frac{\partial x}{\partial \theta} = -\sin\theta \quad \frac{\partial y}{\partial \theta} = \cos\theta$

$\int_C 2y\,dx - x\,dy = \int_0^{2\pi} [2\sin\theta(-\sin\theta) - \cos\theta(\cos\theta)]\,d\theta$

$= \int_0^{2\pi} [-2\sin^2\theta - \cos^2\theta]\,d\theta = \int_0^{2\pi} -[\sin^2\theta + 1]\,d\theta = \int_0^{2\pi} -\left[\frac{1-\cos 2\theta}{2} + 1\right]d\theta$

$= -\int_0^{2\pi} \left[\frac{3}{2} - \frac{\cos 2\theta}{2}\right] d\theta = \left[-\frac{3}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi}$

$= \left[-\frac{3}{2}(2\pi) + 0\right] - [0 + 0] = \boxed{-3\pi}$

b) $\frac{\partial Q}{\partial x} = -1 \quad \frac{\partial P}{\partial y} = 2 \qquad D: \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$\int_C 2y\,dx - x\,dy = \iint_D (-1-2)\,dA = -3\int_0^{2\pi}\int_0^1 r\,dr\,d\theta$

$= -3(2\pi)\left[\frac{r^2}{2}\right]_0^1 = -3(2\pi)\left(\frac{1}{2}\right) = \boxed{-3\pi}$

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

Problem 2. (4)

Find

$$\iiint_E z\, dx dy dz,$$

where E is bounded by the sphere $x^2 + y^2 + z^2 = z$ and the cone $z = \sqrt{x^2 + y^2}$.

$\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta + \rho^2 \cos^2\phi = \rho \cos\phi$

$\Rightarrow \rho^2 \sin^2\phi + \rho^2 \cos^2\phi = \rho\cos\phi$

$\Rightarrow \rho^2 = \rho \cos\phi$

$\Rightarrow \rho = \cos\phi$

$\rho\cos\phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta} = \sqrt{\rho^2 \sin^2\phi} = \rho \sin\phi$

$\Rightarrow \cos\phi = \sin\phi$

$\Rightarrow \phi = \frac{\pi}{4}$

$0 \le \rho \le \cos\phi$
$0 \le \phi \le \frac{\pi}{4}$
$0 \le \theta \le 2\pi$

$$\iiint_E z\, dx dy dz = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho\cos\phi\,(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} \rho^3 \cos\phi \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{\rho^4}{4} \cos\phi \sin\phi \Big|_0^{\cos\phi} d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \frac{\cos^5\phi}{4} \sin\phi\, d\phi\, d\theta = \frac{1}{4}(2\pi)(-1)\left(\frac{1}{6}\right) \cos^6\phi \Big|_0^{\pi/4}$$

$$= -\frac{\pi}{12}\left[\left(\frac{\sqrt{2}}{2}\right)^6 - 1\right] = -\frac{\pi}{12}\left(\frac{8}{64} - 1\right) = -\frac{\pi}{12}\left(\frac{1}{8} - 1\right) = -\frac{\pi}{12}\left(-\frac{7}{8}\right)$$

$$= \boxed{\frac{7\pi}{96}}$$

Problem 3. (4)

Find the line integral

$$\int_C x^3\,dx + y^2\,dy + z\,dz, \qquad \vec{P}_0 \qquad \vec{r}_1$$

where C is the line segment connecting the point $P = (0,0,1)$ and $Q = (1,2,4)$, and C is oriented by the direction from P to Q.

$\vec{r}(t) = \vec{P}_0(1-t) + \vec{P}_1(t)$

$P(t) = \langle 0,0,1\rangle(1-t) + \langle 1,2,4\rangle t = \langle 0,0,1-t\rangle + \langle t, 2t, 4t\rangle$

$= \langle t, 2t, 3t+1\rangle \qquad 0 \le t \le 1$

$\frac{\partial x}{\partial t} = 1 \quad \frac{\partial y}{\partial t} = 2 \quad \frac{\partial z}{\partial t} = 3$

$\int_C x^3 dx + y^2 dy + z dz = \int_0^1 \left[t^3(1) + (2t)^2(2) + (3t+1)(3)\right] dt$

$= \int_0^1 \left[t^3 + 8t^2 + 9t + 3\right] dt$

$= \left[\frac{t^4}{4} + \frac{8}{3}t^3 + \frac{9}{2}t^2 + 3t\right]_0^1 = \frac{1}{4} + \frac{8}{3} + \frac{9}{2} + 3$

$= \frac{3}{12} + \frac{32}{12} + \frac{54}{12} + \frac{36}{12} = \boxed{\frac{125}{12}}$

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Problem 4. (4)

Let $F = (2x + 3y)\mathbf{i} + (3x - 10y)\mathbf{j} + 4z\mathbf{k}$ defined on $\mathbf{R}^3$.
(i) Decide whether or not F is conservative.
(ii) If F is conservative, find the potential function f, such that $F = \nabla f$.
(iii) Compute the line integral $\int_C F \cdot d\mathbf{r}$, where C is given by the parametric equation $x = t$, $y = t^2$, $z = t^3$ $0 \le t \le 1$ with the orientation given by the parametrization.

(i) $\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y & 3x-10y & 4z \end{vmatrix} = (0-0)\vec{i} - (0-0)\vec{j} + (3-3)\vec{k}$

$= 0\vec{i} + 0\vec{j} + 0\vec{k}$ ✓

$\text{Curl}\,\vec{F} = 0 \Rightarrow \boxed{\vec{F} \text{ is conservative}}$

(ii) $\begin{cases} f_x = 2x+3y \\ f_y = 3x-10y \\ f_z = 4z \end{cases}$ $\Rightarrow f = x^2 + 3xy + g(y,z)$

$\Rightarrow f_y = 3x + \frac{\partial g}{\partial y}(y,z) = 3x - 10y$

$\Rightarrow \frac{\partial g}{\partial y}(y,z) = -10y \Rightarrow g(y,z) = -5y^2 + h(z)$

$\Rightarrow f = x^2 + 3xy - 5y^2 + h(z)$

$\Rightarrow f_z = h'(z) = 4z \Rightarrow h(z) = 2z^2 + C$ ✓

$$\boxed{f = x^2 + 3xy - 5y^2 + 2z^2 + C}$$

(iii) $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot dr = f(1) - f(0)$ $\qquad f = t^2 + 3t^3 - 5t^4 + 2t^6 + C$ ✓

$= [1 + 3 - 5 + 2] - [0] = \boxed{1}$

Problem 5. (4)

Let S be a surface given by the parametric equation $\mathbf{r}(u, v) = u\mathbf{i}+v\mathbf{j}+(u^2-2v^2)\mathbf{k}$. Find the equation of the tangent plane of S at the point $(1, 1, -1)$.

$\vec{r}_u = \langle 1, 0, 2u \rangle$

$\vec{r}_v = \langle 0, 1, -4v \rangle$

$$N = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2u \\ 0 & 1 & -4v \end{vmatrix} = (0-2u)\vec{i} - (-4v-0)\vec{j} + (1-0)\vec{k} = \langle -2u, 4v, 1 \rangle$$

At $(1,1,-1)$, $u=1$, $v=1$

At, $u=1, v=1$, $N = \langle -2, 4, 1 \rangle$

$$\boxed{-2(x-1) + 4(y-1) + 1(z+1) = 0}$$