Midterm 2

Name: ______________________________

Student ID: __ __ __ __ __ __ __ __ __

Section:

Tuesday	Thursday
☐ 2A	☐ 2B
☐ 2C	☐ 2D
☐ 2E	☐ 2F

Instructions:

- Do not open this exam until instructed to do so.
- You have 50 minutes to complete the exam.
- Please print your name and student ID number above and check the box of your discussion section.
- **You may not use calculators**, books, notes, or any other material to help you. Please make sure your **phone is silenced and stowed** where you cannot see it.
- We will only grade your work within the pages that are originally included.
- In each of the questions 1 through 5, you must **show your work** to receive full credit. Please write your solutions in the space below the questions. You must indicate if you go over the pages.

Please do not write below this line.

Question	Points	Score
1	12	
2	10	
3	8	
4	10	
5	10	
Total	40	

1. Do the following:

(a) (4 pts) Consider the helix $\mathcal{C}$ parametrized by $\mathbf{r}(t) = \langle 3\cos(t), 3\sin(t), 4t\rangle$ for $0 \le t \le 4\pi$. Compute the length of $\mathcal{C}$.

Solution. Since

$$\|\mathbf{r}'(t)\| = \|\langle -3\sin t, 3\cos t, 4\rangle\| = 5,$$

we have

$$\text{length}(\mathcal{C}) = \int_0^{4\pi} \|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_0^{4\pi} 5\,\mathrm{d}t = \boxed{20\pi}.$$

(b) (4 pts) Let $\mathcal{C}$ be the curve $x^2 - y^2 = 1$ where $x \ge 0$ and $0 \le y \le 2$. Compute the integral of $f(x,y) = 12xy$ over $\mathcal{C}$.

Solution. The equation for the curve can be recast as $x = \sqrt{y^2+1}$. So we may parametrize $\mathcal{C}$ by

$$\mathbf{r}(t) = \langle \sqrt{t^2+1}, t\rangle, \qquad 0 \le t \le 2.$$

Then

$$\mathbf{r}'(t) = \left\langle \frac{t}{\sqrt{t^2+1}}, 1\right\rangle,$$

and so,

$$\int_{\mathcal{C}} f(x,y)\,\mathrm{d}s = \int_0^2 f(\sqrt{t^2+1}, t)\|\mathbf{r}'(t)\|\,\mathrm{d}t = \int_0^2 12t\sqrt{2t^2+1}\,\mathrm{d}t$$
$$= \Big[2(2t^2+1)^{3/2}\Big]_0^2 = \boxed{52}.$$

(c) (4 pts) Integrate $\mathbf{F}(x, y) = \langle x + 3y, x + y\rangle$ over the line segment from $(-1, 1)$ to $(2, 0)$.

Solution. The line segment $\mathcal{C}$ can be parametrized by

$$\mathbf{r}(t) = (1-t)\langle -1, 1\rangle + t\langle 2, 0\rangle = \langle 3t-1, 1-t\rangle, \qquad 0 \le t \le 1.$$

Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \mathbf{F}(3t-1, 1-t) \cdot \langle 3, -1\rangle \,\mathrm{d}t = \int_0^1 (6-2t)\,\mathrm{d}t = \boxed{5}.$$

2. (10 pts) Let $\mathcal{C}$ be the oriented piecewise-linear path in the space joining the points

$$P_0 = (1,0,0), \qquad P_1 = (1,0,1), \qquad P_2 = (1,2,1), \qquad P_3 = (-1,2,1),$$
$$P_4 = (-1,2,-1), \qquad P_5 = (-1,0,1), \qquad P_6 = (0,0,-1)$$

in the order of appearance, as in Figure 1 on the next page. Evaluate

$$\int_{\mathcal{C}} 2xz^3 \, \mathrm{d}x + z^3 \, \mathrm{d}y + (3z^2(x^2+y) - e^{-z}) \, \mathrm{d}z.$$

Note: Justify your work!

Solution. Write $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ for the vector field to be integrated. Then $\mathbf{F}$ satisfies the cross-partial condition:

$$\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x}, \qquad \frac{\partial F_2}{\partial z} = 3z^2 = \frac{\partial F_3}{\partial y}, \qquad \frac{\partial F_3}{\partial x} = 6xz^2 = \frac{\partial F_1}{\partial z}.$$

Since the domain of $\mathbf{F}$ is all of $\mathbb{R}^3$, which is simply connected, we know that $\mathbf{F}$ is conservative. Let f be a potential function of $\mathbf{F}$. Then

$$\frac{\partial f}{\partial x} = 2xz^3 \qquad \Rightarrow \qquad f = \int 2xz^3 \, \mathrm{d}x = x^2z^3 + g(y,z)$$

for some function $g(y,z)$. Then

$$\frac{\partial f}{\partial y} = z^3 \qquad \Rightarrow \qquad \frac{\partial g}{\partial y} = z^3$$
$$\Rightarrow \qquad f = x^2z^3 + \int z^3 \, \mathrm{d}x = (x^2+y)z^3 + h(z)$$

for some function $h(z)$. Finally,

$$\frac{\partial f}{\partial z} = 3z^2(x^2+y) - e^{-z}$$
$$\Rightarrow \qquad h'(z) = -e^{-z}$$
$$\Rightarrow \qquad f = (x^2+y)z^3 + \int (-e^{-z}) \, \mathrm{d}x = (x^2+y)z^3 + e^{-z} + C.$$

Therefore, by the Fundamental Theorem for Conservative Vector Fields,

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = f(P_6) - f(P_0) = \boxed{e-1}.$$

(there is extra working room on the next page)

(extra working room for Problem 2)

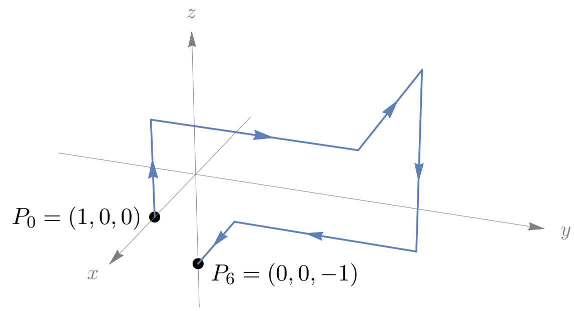

Figure 1. The path $\mathcal{C}$ in Problem 2.

3. Let $\mathbf{F}$ be the vortex field

$$\mathbf{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

which is defined on the plane except at $(0,0)$. Do the following:

(a) (1 pt) Show that $\mathbf{F}(x,y)$ satisfies the cross-partial condition.

> *Solution.* We have
> $$\frac{\partial F_1}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial F_2}{\partial x}.$$

(b) (3 pts) Evaluate $\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$, where $\mathcal{C}$ is the circle $x^2+y^2=4$ oriented counter-clockwise.

Note: Justify your work!

> *Solution.* By parametrizing $\mathcal{C}$ by $\mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle$ for $0 \le t \le 2\pi$, we get
> $$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \left\langle -\frac{2\sin t}{4}, \frac{2\cos t}{4} \right\rangle \cdot \langle -2\sin t, 2\cos t\rangle \, dt = \int_0^{2\pi} dt = \boxed{2\pi}.$$

(c) (4 pts) Evaluate $\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$, where $\mathcal{C}$ is the boundary of the rectangle $[2, 4] \times [-1, 1]$, oriented clockwise.

Note: Provide a full justification for your work, otherwise the point will be scarce.

Solution. Let $\mathcal{D} = \{(x, y) : x > 0\}$ denote the right-half plane. Then

- $\mathbf{F}$ satisfies the cross-partial condition on $\mathcal{D}$, and
- $\mathcal{D}$ is simply connected.

So $\mathbf{F}$ is conservative on $\mathcal{D}$. Moreover, $\mathcal{C}$ lies in $\mathcal{D}$. So

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \boxed{0}$$

by the Fundamental Theorem for Conservative Vector Fields.

4. (10 pts) Let $\mathcal{S}$ be the portion of the sphere $x^2+y^2+z^2=16$, where $4 \le x^2+y^2 \le 8$ and $z \ge 0$. Evaluate

$$\iint_{\mathcal{S}} \frac{1}{z}\,\mathrm{d}S.$$

1st Solution. We parametrize $\mathcal{S}$ by

$$G(r,\theta) = (r\cos\theta, r\sin\theta, \sqrt{16-r^2}), \qquad 2 \le r \le 2\sqrt{2}, \quad 0 \le \theta \le 2\pi.$$

Then

$$\mathbf{T}_r = \left\langle \cos\theta, \sin\theta, -\frac{r}{\sqrt{16-r^2}} \right\rangle, \qquad \mathbf{T}_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle,$$

$$\mathbf{N}(r,\theta) = \left\langle \frac{r^2\cos\theta}{\sqrt{16-r^2}}, \frac{r^2\sin\theta}{\sqrt{16-r^2}}, r \right\rangle, \qquad \|\mathbf{N}(r,\theta)\| = \frac{4r}{\sqrt{16-r^2}}.$$

So the integral is computed as

$$\iint_{\mathcal{S}} \frac{1}{z}\,\mathrm{d}S = \int_0^{2\pi}\int_2^{2\sqrt{2}} \frac{4r}{16-r^2}\,\mathrm{d}r\mathrm{d}\theta = 2\pi\Big[-2\ln(16-r^2)\Big]_{r=2}^{r=2\sqrt{2}} = \boxed{4\pi\ln(3/2)}.$$

2nd Solution. We parametrize $\mathcal{S}$ by

$$G(\theta,\phi) = (4\sin\phi\cos\theta, 4\sin\phi\sin\theta, 4\cos\phi), \qquad 0 \le \theta \le 2\pi, \quad \frac{\pi}{6} \le \phi \le \frac{\pi}{4}.$$

Then we know that

$$\|\mathbf{N}(\theta,\phi)\| = 4^2\sin\phi.$$

So the integral is computed as

$$\iint_{\mathcal{S}} \frac{1}{z}\,\mathrm{d}S = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\int_0^{2\pi} 4\tan\phi\,\mathrm{d}\theta\mathrm{d}\phi = 2\pi\Big[-4\ln\cos\theta\Big]_{\phi=\pi/6}^{\phi=\pi/4} = \boxed{4\pi\ln(3/2)}.$$

5. Let $\mathcal{S}$ be the graph

$$z = \sqrt{x^2 + y^2}, \qquad -2 \le x \le 2, \quad 1 \le y \le 2,$$

which is oriented upward (i.e., the z-component of the normal vector is positive). Do the following:

(a) (4 pts) Find a parametrization $G(u, v)$ for $\mathcal{S}$ and compute $\mathbf{T}_u$, $\mathbf{T}_v$, and $\mathbf{N}(u, v)$.

Solution. We may parametrize $\mathcal{S}$ by

$$G(u, v) = (u, v, \sqrt{u^2 + v^2}), \qquad -2 \le u \le 2, \quad 1 \le v \le 2.$$

Then

$$\mathbf{T}_u = \left\langle 1, 0, \frac{u}{\sqrt{u^2 + v^2}} \right\rangle, \qquad \mathbf{T}_v = \left\langle 0, 1, \frac{v}{\sqrt{u^2 + v^2}} \right\rangle,$$
$$\mathbf{N}(u, v) = \left\langle -\frac{u}{\sqrt{u^2 + v^2}}, -\frac{v}{\sqrt{u^2 + v^2}}, 1 \right\rangle.$$

(b) (2 pts) Consider a vector field $\mathbf{G}$ satisfying $\mathbf{G}(P) = \mathbf{T}_u(P) - 3\mathbf{T}_v(P)$ at every point P of $\mathcal{S}$. Evaluate $\iint_{\mathcal{S}} \mathbf{G} \cdot d\mathbf{S}$.

Hint: Recall the definition of the vector surface integral.

Solution. If $\mathbf{n}$ denotes the unit normal vector field for the orientation of $\mathcal{S}$, then

$$\iint_{\mathcal{S}} \mathbf{G} \cdot d\mathbf{S} = \iint_{\mathcal{S}} (\mathbf{G} \cdot \mathbf{n})\, dS.$$

But since $\mathbf{n}$ is normal to both tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$, we must have $\mathbf{G} \cdot \mathbf{n} = 0$. Therefore

$$\iint_{\mathcal{S}} \mathbf{G} \cdot d\mathbf{S} = \boxed{0}.$$

(c) (4 pts) Evaluate $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = \langle ze^y, 3yz, 2\rangle$.

Solution. We have

$$\mathbf{F}(G(u,v)) \cdot \mathbf{N}(u,v) = -ue^v - 3v^2 + 2.$$

So it follows that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \int_1^2 \int_{-1}^1 (2 - ue^v - 3v^2)\, dudv = \int_1^2 (8 - 12v^2)\, dv = \boxed{-20}.$$

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