32B Lect. 3 | Midterm 2 | February 28

First Name: [redacted] ID#[redacted]

Last Name: [redacted]

Section: [redacted] $= \begin{cases} 3a & \text{Tuesday with Allen Boozer} \\ 3b & \text{Thursday with Allen Boozer} \\ 3c & \text{Tuesday with Steven Gagniere} \\ 3d & \text{Thursday with Steven Gagniere} \\ 3e & \text{Tuesday with Francis White} \\ 3f & \text{Thursday with Francis White} \end{cases}$

Rules.

- There are **FOUR** problems; fifteen points per problem.
- There are two extra pages at the end. You may also use the backs of pages.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone.

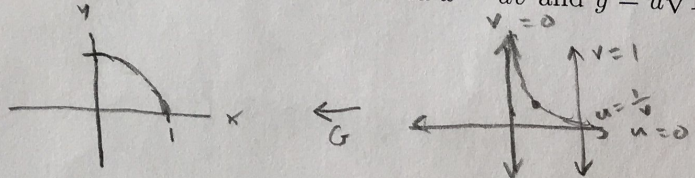
1	2	3	4	Σ
5	10	5	6	26

5

(1) Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} xy^3 \, dy \, dx$$

by changing variables via $x = uv$ and $y = u\sqrt{1-v^2}$.



$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1$

$G(u,v) = (uv, u\sqrt{1-v^2}) = (x,y)$

$x = 0 \to uv = 0 \to u = 0$ or $v = 0$

$y = 0 \to u\sqrt{1-v^2} = 0 \to u = 0$ or $v = \pm 1$ $\quad v = 1$

$x^2 + y^2 = 1 \to u^2v^2 + u^2(1-v^2) = 0 \to u = 0$

$\cancel{u^2v^2} + u^2 \cancel{-u^2v^2} = 0$

$y = \sqrt{1-x^2}$

$x = 1 \to uv = 1 \to u = \frac{1}{v}$

$$\iint_D (uv)(u\sqrt{1-v^2})^3 \, |\mathrm{Jac}(G)| \, du\, dv$$

abs value

$$= \int_0^1 \int_0^{1/v} u^4 v (1-v^2)^{3/2} \left(\cancel{\frac{u}{(1-v^2)^{1/2}}} \right) du\, dv$$

$$= \int_0^1 \int_0^{1/v} u^5 v (1-v^2)\, du\, dv$$

$$= \int_0^1 \frac{u^6}{6} v(1-v^2) \Big|_0^{1/v} dv$$

$$= \frac{1}{6}\int_0^1 \frac{(1-v^2)}{v^5}\, dv = \frac{1}{6}\int_0^1 v^{-5} - v^{-3}\, dv$$

$$= \frac{1}{6}\left(-\frac{1}{4}v^{-4} + \frac{1}{2}v^{-2}\right)\Big|_0^1 = -\frac{1}{24} + \frac{1}{12} = \cancel{\boxed{\frac{1}{12}}}$$

$$\mathrm{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} v & u \\ \sqrt{1-v^2} & -\frac{uv}{\sqrt{1-v^2}} \end{vmatrix}$$

$$\frac{\partial}{\partial v}\left(u\sqrt{1-v^2}\right)$$

$$= u\left(\frac{1}{2}(1-v^2)^{-\frac{1}{2}}(-\cancel{2}v)\right)$$

$$= -\frac{uv}{\sqrt{1-v^2}}$$

$$\mathrm{Jac}(G) = -\frac{uv^2}{\sqrt{1-v^2}} + u\sqrt{1-v^2}$$

$$= \frac{-uv^2 - u(1-v^2)}{\sqrt{1-v^2}} = \frac{-u}{\sqrt{1-v^2}}$$

(2) Compute

$$\int_\gamma \sqrt{x^2 + 9y^2}\, ds$$

where γ passes from $(0,0)$ to $(1,1)$ along the curve $y = x^3$.

$f = \sqrt{x^2 + 9y^2}$

$f(\vec{r}(t)) = \sqrt{t^2 + 9t^6} = t\sqrt{1+9t^4}$ ✓

$\vec{r}(t) = \begin{pmatrix} t \\ t^3 \end{pmatrix} \quad 0 \le t \le 1$

$\int_0^1 f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt$ ✓

$\vec{r}\,'(t) = \begin{pmatrix} 1 \\ 3t^2 \end{pmatrix} \quad \|\vec{r}\,'(t)\| = \sqrt{1+9t^4}$ ✓

$= \int_0^1 \left(t\sqrt{1+9t^4}\right)\left(\sqrt{1+9t^4}\right) dt$

$= \int_0^1 t(1+9t^4)\, dt = \int_0^1 t + 9t^5\, dt = \frac{1}{2}t^2 + \frac{3}{2}t^6 \Big|_0^1 = \boxed{2}$ ✓

2

(3) (a) For what value of λ is the vector field

$$\vec{F}(x,y) = \begin{bmatrix} \dfrac{-y^2}{(x+1)^2} \\ \dfrac{\lambda y}{x+1} \end{bmatrix} \begin{array}{l} P \\ Q \end{array} \qquad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

cross partials works if conservative

conservative in the region where $x > -1$?
(b) Find a potential with this value of λ.

(A) $\dfrac{-2y}{(x+1)^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\lambda y}{x+1}\right)$

$= \dfrac{\partial}{\partial x}\left(y\lambda(x+1)^{-1}\right)$

$= y\dfrac{\partial}{\partial x}\lambda(x+1)^{-1}$

$= -\dfrac{y}{(x+1)^2}\dfrac{\partial}{\partial x}\lambda$

$\therefore \boxed{\lambda = 2x + C}$

3

$$\nabla f = \begin{pmatrix} \dfrac{-y^2}{(x+1)^2} \\ \dfrac{-2xy}{x+1} \end{pmatrix}$$

$\dfrac{\partial f}{\partial x} = \dfrac{-y^2}{(x+1)^2} \rightarrow f = y^2(x+1)^{-1} + \ldots + C_2$

$\dfrac{\partial f}{\partial y} = -\dfrac{2xy + cy}{x+1} \rightarrow f = -\dfrac{1}{2}y^2\left(\dfrac{2x+c}{x+1}\right) + \ldots + C_2$

$-(2xy + cy)(x+1)^{-1}$

$-(2xy + cy)$

$f(x,y) = \dfrac{y^2}{x+1} + \dfrac{(x)y^2}{x+1} = \dfrac{y^2(1+x+C)}{x+1}$

$$\boxed{f(x,y) = y^2\left(1 + \frac{C_1}{x+1}\right) + C_2}$$

z can't be zero

only true if half of two sheet hyperboloid

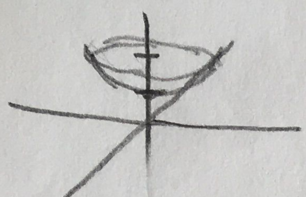
convert to spherical / polar?

(4) Let $\mathcal{H}$ denote the surface where $z^2 = 1 + x^2 + y^2$ and $0 \le z \le 2$. Compute

$$\int_{\mathcal{H}} z \, dS$$

really $1 \le z \le 2$

f



min of $z = 1$

$z = \sqrt{1+x^2+y^2}$ since $z \ge 0$

$$\iint_D f(G(x,y)) \left\| \frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} \right\| \, dx\, dy$$

$G(x,y) = \begin{pmatrix} x \\ y \\ \sqrt{1+x^2+y^2} \end{pmatrix}$ ✓

$$\left\| \frac{\partial G}{\partial x} \times \frac{\partial G}{\partial y} \right\| = \left\| \begin{pmatrix} 1 \\ 0 \\ \frac{x}{\sqrt{1+x^2+y^2}} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{y}{\sqrt{1+x^2+y^2}} \end{pmatrix} \right\|$$

$$= \begin{pmatrix} 0 - \frac{x}{\sqrt{1+x^2+y^2}} \\ \frac{y}{\sqrt{1+x^2+y^2}} - 0 \\ 1 - 0 \end{pmatrix}$$

$$\| G(x,y) \| = \sqrt{\frac{x^2+y^2}{1+x^2+y^2} + 1}$$ ✓

$$= \iint_D \sqrt{1+x^2+y^2} \sqrt{\frac{x^2+y^2}{1+x^2+y^2} + 1} \, dx\, dy$$

$$= \iint_D \sqrt{\cancel{(1+x^2+y^2)} \left(\frac{x^2+y^2+(1+x^2+y^2)}{\cancel{1+x^2+y^2}} \right)} \, dx\, dy$$

Jacobian

$x = r\cos\theta$

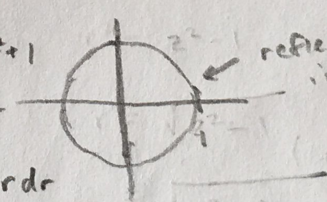
$y = r\sin\theta$

$$= \int_0^{2\pi} \int_0^{\cancel{1}} \sqrt{2r^2+1} \; r \, dr\, d\theta$$ -2

$u = 2r^2+1$

$du = 4r\,dr$

$\frac{1}{4} du = r\,dr$



reflection is circle

$$= \frac{1}{4} \int_0^{2\pi} \int_1^3 u^{1/2} \, du\, d\theta$$

$$= \frac{1}{4}(2\pi) \int_1^3 u^{1/2} du = \frac{\pi}{2} \left(\frac{2}{3} u^{3/2} \right) \Big|_1^3$$

$$= \frac{\pi}{3}(3\sqrt{3} - 1) = \boxed{\left(\sqrt{3} - \frac{1}{3}\right)\pi}$$ -2