32B Lect. 3 Midterm 1 January 31

First Name: [redacted] ID# [redacted]

Last Name: [redacted]

Section: [redacted] $= \begin{cases} 3a & \text{Tuesday with Allen Boozer} \\ 3b & \text{Thursday with Allen Boozer} \\ 3c & \text{Tuesday with Steven Gagniere} \\ 3d & \text{Thursday with Steven Gagniere} \\ 3e & \text{Tuesday with Francis White} \\ 3f & \text{Thursday with Francis White} \end{cases}$

Rules.

- There are **FOUR** problems; ten points per problem.
- There is an extra page at the end. You may also use the backs of pages.
- No calculators, computers, notes, books, crib-sheets,...
- Out of consideration for your class-mates, no chewing, humming, pen-twirling, snoring,... Try to sit still.
- Turn off your cell-phone, pager,...

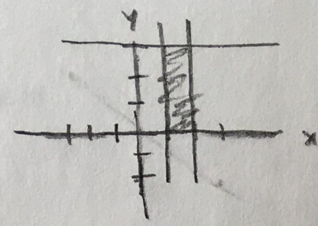
1	2	3	4	Σ
10	10	9	7	36

(1) Determine the average value of

$2y = -3x$

$y = -\frac{3}{2}x$

$$f(x, y) = 3x + 2y$$



over the region where $1 \leq x \leq 2$ and $0 \leq y \leq 3$.

$$\bar{f} = \frac{\iint f(x,y)\, dA}{\iint 1\, dA} = \frac{\int_0^3 \int_1^2 3x + 2y \; dx\, dy}{\int_0^3 \int_1^2 1 \; dx\, dy} = \frac{\int_0^3 \frac{3}{2}x^2 + 2yx \Big|_1^2 \, dy}{\int_0^3 x \Big|_1^2 \, dy}$$

volume

$$= \frac{\int_0^3 6 + 4y - \left(\frac{3}{2} + 2y\right) dy}{\int_0^3 1 \; dy} = \frac{\int_0^3 \frac{9}{2} + 2y \; dy}{3} = \frac{\frac{9}{2}y + y^2 \Big|_0^3}{3} = \frac{\frac{27}{2} + 9}{3}$$

13.5

$$= \frac{\frac{27 + 18}{2}}{3} = \frac{45}{6} = \boxed{\frac{15}{2}}$$

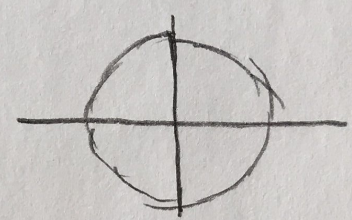
10

(2) The joint probability density function of X and Y is given by

$$p(x,y) = \frac{e^{-(x^2+y^2)/2}}{2\pi}$$

where (x, y) ranges over the entire plane. Determine the median of $X^2 + Y^2$.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{\frac{-(x^2+y^2)}{2}}}{2\pi}\, dA = 1$$



$\text{Prob}(X^2+Y^2 < m) = \frac{1}{2}$ Find m.

find an $r^2 = m$ where density inside circle $= \frac{1}{2}$

$y^2 < m - x^2$

$X^2+Y^2 = r^2 < m$

$Y < \pm\sqrt{m-x^2}$

$\text{Prob}(r^2 < m) = \frac{1}{2}$

$r = \sqrt{m}$

Jacobian factor

polar coordinates

$$\int_0^{2\pi}\int_0^{\sqrt{m}} \frac{e^{-\frac{1}{2}r^2}}{2\pi}\left(r\, dr\, d\theta\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\sqrt{m}} re^{-\frac{1}{2}r^2}\, dr\, d\theta$$

$u = -\frac{1}{2}r^2$

$du = -r\, dr$

$-du = r\, dr$

$$= -\frac{1}{2\pi}\int_0^{2\pi}\int_0^{-\frac{m}{2}} e^u\, du\, d\theta$$

$-\frac{1}{2}(\sqrt{m})^2 = -\frac{1}{2}m$

$$= -\frac{1}{2\pi}\int_0^{2\pi} e^{-\frac{m}{2}} - 1\, d\theta$$

$$= -\frac{\cancel{2\pi}}{\cancel{2\pi}}\left(e^{-\frac{m}{2}} - 1\right) = 1 - e^{-\frac{m}{2}} = \frac{1}{2}$$

solve for m while prob. integral $= \frac{1}{2}$

$$\frac{1}{2} = \frac{1}{e^{m/2}}$$

$2 = e^{\frac{1}{2}m}$

$\ln 2 = \frac{1}{2}m$

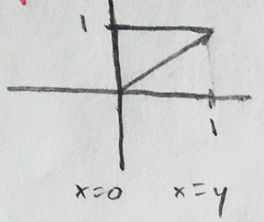
$$\boxed{m = 2\ln 2}$$

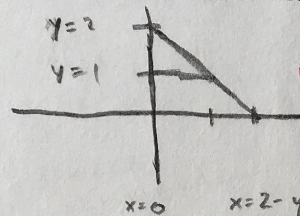
(3) Evaluate the following by reversing the order of integration.

$$\int_0^1 \int_0^y \sin\left((x-1)^2\right)\, dx\, dy + \int_1^2 \int_0^{2-y} \sin\left((x-1)^2\right)\, dx\, dy$$

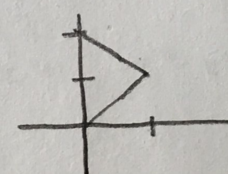
9

Hint: Reversing the order will allow you to combine the two regions into one.





y = -x + 2



$$\int_0^1 \int_x^1 \sin((x-1)^2)\, dy\, dx + \int_0^1 \int_1^{2-x} \sin((x-1)^2)\, dy\, dx$$

$$= \int_0^1 \int_x^{2-x} \sin((x-1)^2)\, dy\, dx = \int_0^1 \sin((x-1)^2)(2-x-x)\, dx$$

$u = (x-1)^2$

$du = 2(x-1)dx$

$$= \int_0^1 \sin((x-1)^2)(2-2x)\, dx = -2\int_0^1 \sin((x-1)^2)(x-1)\, dx$$

$u = (x-1)^2$

$\frac{1}{2} du = (x-1)dx$

$$= -\cancel{2}\left(\tfrac{1}{2}\right)\int_1^0 \sin(u)\, du = -2\int_0^1 \sin u\, du$$

$(1-1)^2 = 0$

$(0-1)^2 = 1$

$$= 2(-\cos u)\Big|_0^1 = 2(-\cos(1) + \cos(0))$$

1

$$= \boxed{2 - 2\cos(1)}$$

(4) Determine the volume of the region defined by the following inequalities

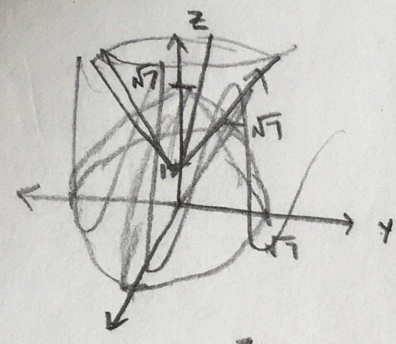
$$z \geq 0, \qquad x^2 + y^2 \leq z^2 + 1, \qquad x^2 + y^2 + z^2 \leq 7$$

one-sheet hyperbola but only top half / not exactly cone

$z \geq 0$

sphere

use spherical coordinates

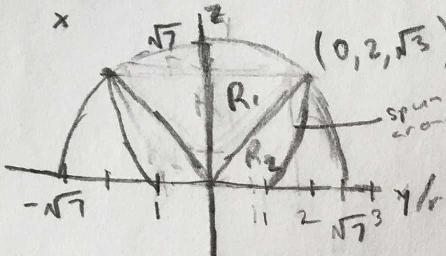


$x^2+y^2 = z^2+1$

$r = \sqrt{7}$

least it can be is 1

gonna be raised one



x = 0 spun around cross-section

$r^2 = z^2+1$ $\qquad r^2+z^2=7$

$y^2 = z^2+1 \qquad y^2+z^2 = 7$

$y^2 = 7 - z^2$

$7 - z^2 = z^2+1$

$r^2 = z^2+1$

$6 = 2z^2$

$1 \to \sqrt{3}$

$z^2 = 3$

$0 \to 2$

$y = \sqrt{4}$

$z = \sqrt{3}$

$\sin\left(\frac{\pi}{2}\right) = 1$

$\cos\left(\frac{\pi}{2}\right) = 0$

$r^2 \leq z^2+1$

$z = 0 \to r = 1$

r cannot be 0

$r^2 = z^2+1 \qquad r^2+z^2 = 7$

$z^2 = r^2-1$

$2r^2 - 1 = 7$

$z = \sqrt{1} \to r = \sqrt{2}$

$z = \sqrt{2} \to r = \sqrt{3}$

$z = \sqrt{3} \to r = 2 = \sqrt{4}$

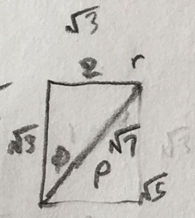
$2r^2 = 8$

where they intersect

$r = 2 \;\therefore$

$z = \sqrt{3}$

use three integrals



$2\sqrt{2}$

$\tan\phi = \frac{2}{\sqrt{3}}$

$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$\cos^2\phi + \sin^2\phi = 1$

$\cos^2\phi = 1 - \sin^2\phi$

$z = \rho\cos\phi$

$\phi = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$r = \rho\sin\phi$

$$\int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)}\int_0^{\sqrt{7}} \rho^2\sin\phi \, d\rho\, d\phi\, d\theta + \int_0^{2\pi}\int_{\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)}^{\pi/2}\int_0^{\frac{1}{\sqrt{\sin^2\phi-\cos^2\phi}}} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$r^2 = z^2+1$

$1 - \sin^2\phi$

$\rho^2\sin^2\phi = \rho^2\cos^2\phi + 1$

$$= 2\pi\left(\int_0^{\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)} \frac{7^{3/2}}{3}\sin\phi\, d\phi + \int_{\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)}^{\pi/2} \frac{(\sin^2\phi - \cos^2\phi)^{3/2}}{3}\, d\phi\right) d\theta$$

$\rho^2(\sin^2\phi - \cos^2\phi) = 1$

$2\sin^2\phi - 1 = (\cos(2\phi))^{3/2}$?

this one

$u = 2\phi$

$du = 2\,d\phi$

$\rho = \frac{1}{\sqrt{\sin^2\phi - \cos^2\phi}}$

$\frac{1}{2}du = d\phi$

$$-\frac{7^{3/2}}{3}\cos\phi \Big|_0^{\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)} + \int_{\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)}^{\pi/2} \frac{(\cos(2\phi))^{3/2}}{3}\, d\phi$$

$\int^{\pi} \frac{\cos u}{3}$

$$-\frac{(\sqrt{7})^3}{3}\left(\frac{\sqrt{3}}{\sqrt{7}}\right) \to \boxed{2\pi\left(\frac{7^{3/2} - 7\sqrt{3}}{3} + \int_{\tan^{-1}(2/\sqrt{3})}^{\pi/2} \sim\right)}$$