Math 32B, Lecture 4
Multivariable Calculus

Sample Final Exam

Instructions: You have three hours to complete the exam. There are ten problems, worth a total of one hundred points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Name: ______________________________

UID: ______________________________

Section: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

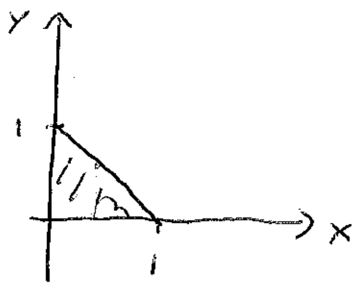
Problem 1.
Let

$$p(x,y) = \begin{cases} Cxy \text{ if } 0 \le x \le 1 \text{ and } 0 \le y \le 1-x \\ 0 \quad \text{otherwise} \end{cases}$$

(a) [5pts.] Find a constant C that makes $p(x,y)$ into a probability distribution.

(b) [5pts.] Find $P(X \geq Y)$.

(a) We want the total integral over the plane to be 1.



$p(x,y)$ is nonzero on the triangle shown

$$\int_0^1 \int_0^{1-x} (Cxy)\,dy\,dx$$

$$= \int_0^1 Cx \left(\tfrac{1}{2}y^2\right)\Big|_0^{1-x} dx$$

$$= \frac{C}{2}\int_0^1 x(1-2x+x^2)\,dx$$

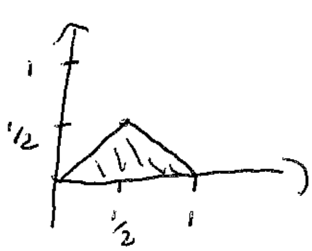
$$= \frac{C}{2}\left[\int_0^1 (x - 2x^2 + x^3)\,dx\right.$$

$$= \frac{C}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right]$$

$$= \frac{C}{2}\left(\frac{1}{12}\right)$$

$$= \frac{C}{24} \qquad \text{So} \quad C = 24$$

(b) The region where $x \geq y$ and $p(x,y)$ is nonzero is the triangle



We integrate $\displaystyle\int_0^{1/2}\int_y^{1-y} 24xy\,dx\,dy$

ctd

$$= \int_0^{1/2} \left(12yx^2 \Big|_y^{1-y} \right) dy$$

$$= 12 \int_0^{1/2} y \left[(1-y)^2 - y^2 \right] dy$$

$$= 12 \int_0^{1/2} (y - 2y^2)\, dy$$

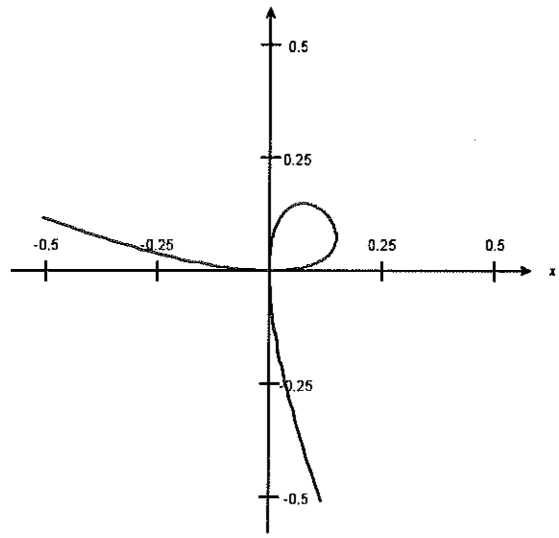
$$= 12 \left[\frac{1}{2}y^2 - \frac{2}{3}y^3 \right]_0^{1/2}$$

$$= 6\left(\frac{1}{4}\right) - 8\left(\frac{1}{8}\right)$$

$$= \frac{1}{2}$$

Problem 2.

Let $\mathbf{r}(t) = \langle t^2(1-t), t(t-1)^2 \rangle$. A plot of $\mathbf{r}(t)$ is shown below.



(a) [5pts.] Compute the area enclosed by the loop in the curve.

(b) [5pts.] What is the flux of the vector field $\mathbf{F}(x, y) = \langle 2x - 7y^2, 9x - 2y \rangle$ out of the loop?

Note that we run clockwise around the loop on $0 \le t \le 1$

ⓐ $\vec{r}(t) = \langle t^2 - t^3, t^3 - 2t^2 + t \rangle$

$\vec{r}\,'(t) = \langle 2t - 3t^2, 3t^2 - 4t + 1 \rangle$

$$\text{Area} = -\int_{\mathcal{C}} x\,dy$$

$$= -\int_0^1 (t^2 - t^3)(3t^2 - 4t + 1)\,dt$$

$$= -\int_0^1 [3t^4 - 4t^3 + t^2 - 3t^5 + 4t^4 - t^3]\,dt$$

$$= -\int_0^1 [7t^4 - 5t^3 + t^2 - 3t^5]\,dt$$

$$= -\left[\frac{7}{5} - \frac{5}{4} + \frac{1}{3} - \frac{3}{6}\right]$$

$$= \frac{1}{60}$$

ⓑ $\text{Flux} = \int_{\mathcal{C}} \vec{F} \cdot \vec{n}\,ds$

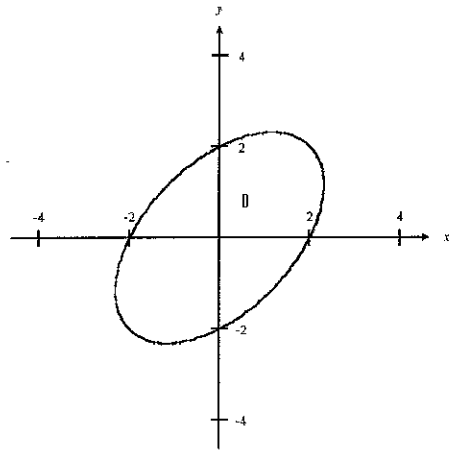
$$= \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dA$$

$$= \iint_D (2 - 2)\,dA$$

$$= 0$$

Problem 3.

Consider the domain $\mathcal{D}$ shown below, which consists of the points x, y such that $x^2 - xy + y^2 \leq 4$.



(a) [5pts.] Suppose that $G(u,v) = \left(2u - \frac{2}{\sqrt{3}}v, 2u + \frac{2}{\sqrt{3}}v\right)$. Find the region in the uv-plane that maps to $\mathcal{D}$ under G.

(b) [5pts.] What is $\int\int_{\mathcal{D}}(x^2 - xy + y^2)dA$?

ⓐ Notice that

$$x^2 - xy + y^2 = \left(4u^2 - \cancel{\frac{8uv}{\sqrt{3}}} + \frac{4}{3}v^2\right) - \left(4u^2 - \frac{4}{3}v^2\right) + \left(4u^2 + \cancel{\frac{8uv}{\sqrt{3}}} + \frac{4}{3}v^2\right)$$

$$= 4u^2 + 3\left(\frac{4}{3}v^2\right)$$

$$= 4(u^2+v^2)$$

So $x^2 - xy + y^2 \leq 4$ is the region $u^2+v^2 \leq 1$, a unit disk in the uv-plane. Call this disk $\mathcal{D}_0$.

ⓑ Jacobian $\begin{vmatrix} 2 & \frac{-2}{\sqrt{3}} \\ 2 & \frac{2}{\sqrt{3}} \end{vmatrix} = \frac{4}{\sqrt{3}} - \frac{-4}{\sqrt{3}} = \frac{8}{\sqrt{3}}$

$$\iint_{\mathcal{D}} (x^2-xy+y^2)\,dA = \iint_{\mathcal{D}_0} 4(u^2+v^2)\left(\frac{8}{\sqrt{3}}\right) = \int_0^{2\pi}\int_0^1 4r^2\left(\frac{8}{\sqrt{3}}\right) r\,dr\,d\theta$$

ctd

$$= 2\pi\left(\frac{32}{\sqrt{3}}\right)\left[\frac{1}{4}r^4\right]_0^1$$

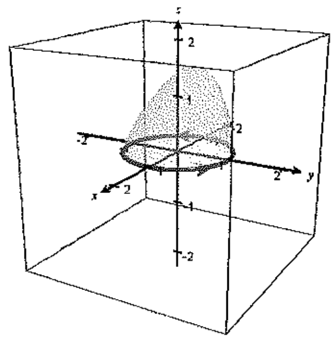
$$= \frac{16\pi}{\sqrt{3}}$$

Problem 4.

Consider the vector field $\mathbf{F} = \langle 2ye^z - xy, y, yz - z \rangle$.

(a) [5pts.] Verify that $\mathbf{A} = \langle yz, xyz, y^2e^z \rangle$ is a vector potential for $\mathbf{F}$.

(b) [5pts.] What is the flux of $\mathbf{F}$ across the surface shown? The marked curve is the boundary of the surface.



(a) $\text{curl}(\vec{A}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xyz & y^2e^z \end{vmatrix} = \langle 2ye^z - xy, +(0+y), yz - z \rangle$ ✓

(b) Use Stokes thm

$\iint_S \vec{F} \cdot d\vec{S} = \int_C \vec{A} \cdot d\vec{r}$

$= \int_C 0 \, d\vec{r}$

$= 0$

$\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$

$\vec{A}(\vec{r}(t)) = \langle 0, 0, \sin^2 t \rangle$

Problem 5.

Consider the vector field $\mathbf{F}(x,y) = \langle 9y - y^3, e^{\sqrt{y}}(x^2 - 3x)\rangle$. Let C be the square with corners $(0,0)$, $(0,3)$, $(3,3)$, and $(0,3)$, oriented counterclockwise.

(a) [5pts.] Show that $\mathbf{F}$ is not conservative.

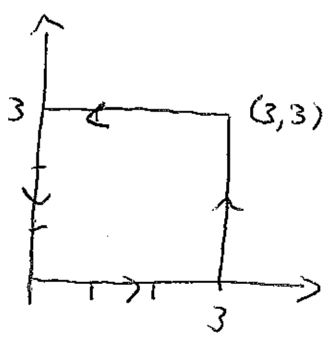
(b) [5pts.] What is $\oint_C \mathbf{F} \cdot d\mathbf{r}$? [Hint: What does $\mathbf{F}$ look like on C?]

(a) $\dfrac{\partial F_1}{\partial y} = 9 - 3y^2$

Not equal $\Rightarrow$ not conservative.

$\dfrac{\partial F_2}{\partial x} = e^{\sqrt{y}}(2x-3)$

(b)



On the top edge, $y = 3$

$\vec{F} = \langle 9(3) - 27, e^{\sqrt{3}}(x^2 - 3x)\rangle$

$= \langle 0, e^{\sqrt{3}}(x^2 - 3x)\rangle$

Note $\perp$ to $\vec{r}\,'(t) = \langle -1, 0\rangle$

On the left edge, $x = 0$.

$\vec{F} = \langle 9y - y^3, 0\rangle$

Note $\perp$ to $\vec{r}\,'(t) = \langle 0, -1\rangle$

On the bottom edge, $y = 0$. $\vec{F} = \langle 0, x^2 - 3x\rangle$. Note $\perp$ to $\vec{r}\,'(t) = \langle 1, 0\rangle$

On the right edge, $x = 3$, $\vec{F} = \langle 9y - y^3, e^{\sqrt{y}}(9 - 3(3))\rangle = \langle 9y - y^3, 0\rangle$.

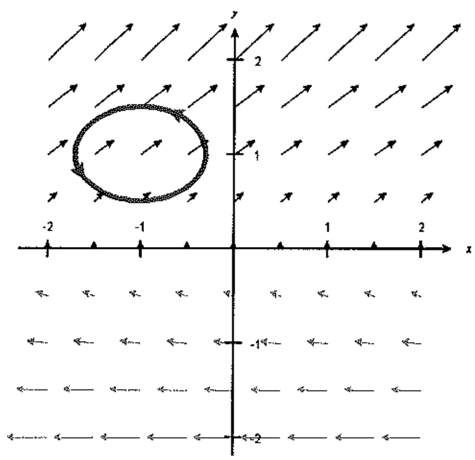
Note $\perp$ to $\vec{r}\,'(t) = \langle 0, 1\rangle$.

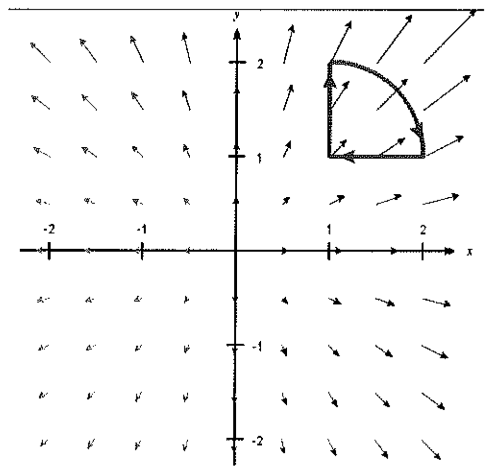
We see $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = 0$ for every edge of the square w/

$\int_e \vec{F} \cdot d\vec{r} = 0$.

Problem 6.

Consider the vector fields and paths shown below. The path on the left is C_1 and the path on the right is C_2.





(a) [5pts.] For each vector field above, decide whether curl($\mathbf{F}$) is positive, negative, or zero at the origin. Justify your answers.

(b) [5pts.] Decide whether the line integrals $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$ and $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r}$ are positive, negative, or zero. Justify your answers.

(a) Imagine placing a wheel at the origin. On the left, three vectors push it clockwise and one counterclockwise, and they have roughly equal magnitude, so it turns clockwise $\rightsquigarrow$ curl is negative. On the right, all vectors push outward and the wheel does not turn $\rightsquigarrow$ curl is 0.

(b) $\oint_{C_1} \vec{F} \cdot d\vec{r}$ is negative. $\vec{r}(t)$ makes acute angles with $\vec{F}$ along the bottom edge of the ellipse (positive contribution) but obtuse angles w/ vectors of F higher magnitude along the top edge (larger negative contribution).

$\oint_{C_2} \vec{F} \cdot d\vec{r}$ is 0. $\vec{F}$ is orthogonal to $\vec{r}\,'(t)$ on the circular edge (contribution zero) and contributions from the two straight edges exactly cancel.

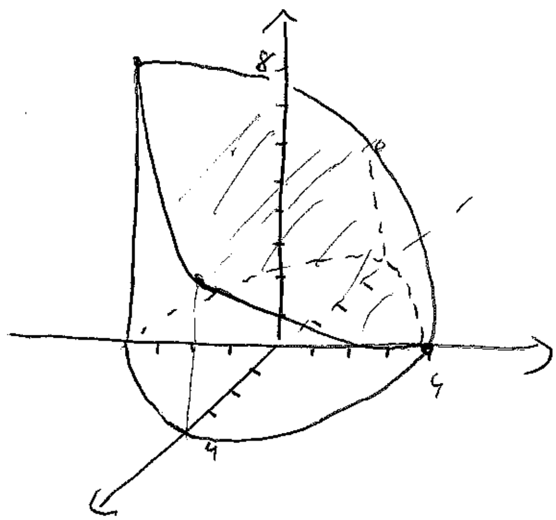
Problem 7.

Let S be the boundary of the solid bounded by the cylinder $x^2+y^2=16$ and the planes $z=0$ and $z=4-y$, oriented outward.

(a) [5pts.] Draw this surface.

(b) [5pts.] Find the flux of the vector field $\mathbf{F}(x,y,z) = \langle xyz+xy, \frac{1}{2}y^2(1-z)+e^x, e^{x^2+y^2}\rangle$ through S.

(a)



(b) Use divergence thm on W w/ $\partial W = S$.

$$\operatorname{div}\vec{F} = (yz+y) + (y-yz) + 0 = 2y$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^4\int_0^{4-r\sin\theta} 2r\sin\theta \, r \, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 (4-r\sin\theta)\, 2r^2\sin\theta \, dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 (\cancel{8r^2\sin\theta}^{\,0} - 2r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{1}{2}(4)^4\sin^2\theta \, d\theta$$

$$= -128\pi$$

Problem 8.

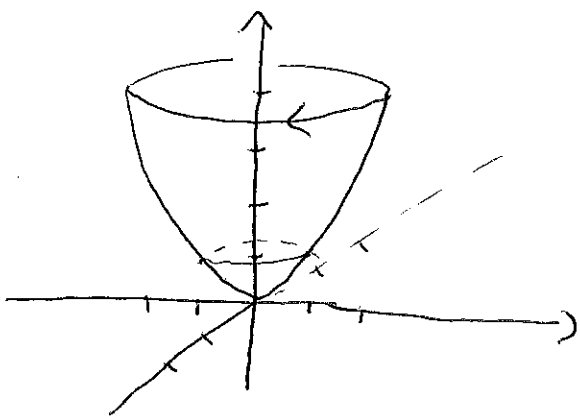
Consider the surface $z = x^2 + y^2$ with $0 \le z \le 4$, with the outward pointing normal vector.

(a) [3pts.] Draw this surface. Be sure to orient the boundary.

(b) [3pts.] Is the flux of $\mathbf{F}(x, y, z) = \langle 2x, 0, -7z^2 \rangle$ across $\mathcal{S}$ positive or negative? Justify your answer.

(c) [4pts.] Find the surface area of $\mathcal{S}$. Use any method you like.

(a)



(b) Notice that $\vec{F}(x,y,z)$ points outward and downward, as does the unit normal.

Indeed, $\vec{N} = \langle 2x, 2y, -1 \rangle$ is the normal vector to the parametrization $G(x,y) = (x, y, x^2+y^2)$, so

$$\text{Flux} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_D \vec{F} \cdot \vec{N}\, dA = \iint_D \underbrace{(4x^2 + 7z^2)}_{\text{Positive}}\, dA > 0.$$

(c) We see that $\|\vec{N}\| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4r^2+1}$. We integrate over the circle of radius 2:

$$\text{Area} = \iint_D \|\vec{N}\|\, dA$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{4r^2+1}\; r\,dr\,d\theta \quad \text{etc}$$

$$= 2\pi\left(\frac{1}{8}\right)\left(\frac{2}{3}\right)(4r^2+1)^{3/2}\Big|_0^2$$

$$= \frac{\pi}{6}\left[17^{3/2} - 1\right]$$