Math 32B, Lecture 4
Multivariable Calculus

Midterm 2

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Name: ______________________________

UID: ______________________________

Section: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

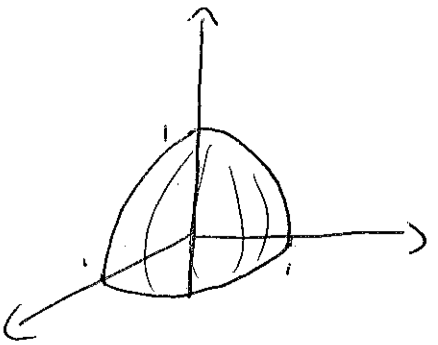
Problem 1.

Consider the solid consisting of the portion of the unit ball $x^2+y^2+z^2=1$ lying in the first octant, with mass density $\delta(x,y,z)=x$.

(a) [5pts.] What is the mass of this solid?

(b) [5pts.] What is the y-coordinate of the center of mass of this solid?

(a) We use spherical coordinates.



$$x = \rho\cos\theta\sin\phi$$

$$M = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho\cos\theta\sin\phi\,(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta)$$

$$= \int_0^{\pi/2}\cos\theta\,d\theta \cdot \int_0^1 \rho^3\,d\rho \cdot \int_0^{\pi/2}\sin^2\phi\,d\phi$$

$$= \sin\theta\Big|_0^{\pi/2} \cdot \frac{1}{4}\rho^4\Big|_0^1 \cdot \frac{1}{2}\left[1-\frac{1}{2}\sin(2\phi)\right]_0^{\pi/2}$$

$$= 1\cdot\frac{1}{4}\cdot\frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{16}$$

(b) $M_{xz} = \iiint_W yx\,dV$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\sin\theta\sin\phi)(\rho\cos\theta\sin\phi)(\rho^2\sin\phi\,d\rho\,d\phi\,d\theta)$$

$$= \int_0^{\pi/2}\sin\theta\cos\theta\,d\theta \cdot \int_0^1\rho^4\,d\rho \cdot \int_0^{\pi/2}\sin^3\phi\,d\phi$$

$$= \frac{1}{2}\sin^2\theta\Big|_0^{\pi/2} \cdot \frac{1}{5}\rho^5\Big|_0^1 \cdot \int_0^{\pi/2}\sin\phi(1-\cos^2\phi)\,d\phi$$

$$= \frac{1}{2}\cdot\frac{1}{5}\cdot\left[-\cos\phi+\frac{1}{3}\cos^3\phi\right]_0^{\pi/2} = \frac{1}{10}\left(\frac{2}{3}\right) = \frac{1}{15}$$

$$= \frac{1}{10}\left[(0)-\left(-1+\frac{1}{3}\right)\right]$$

y-coordinate of center of mass $= \frac{16}{15\pi}$

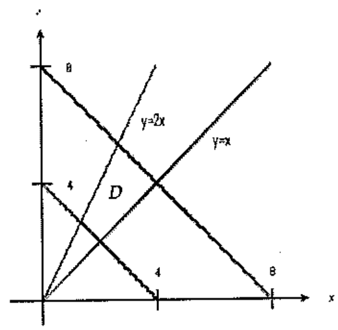
Problem 2.

Consider the map

$$G(u,v) = \left(\frac{u}{v+1}, \frac{uv}{v+1}\right)$$

from the uv-plane to the xy-plane.

(a) [5pts.] Find a domain on the uv-plane that maps to the domain $\mathcal{D}$ on the xy-plane pictured here.



(b) [5pts.] Use your answer from part (a) to compute $\int\int_{\mathcal{D}}(x+y)dxdy$.

(a) We find the preimages of the boundary lines.

$y=x$: $\frac{uv}{v+1} = \frac{u}{v+1} \rightsquigarrow v=1$ (or $u=0$)

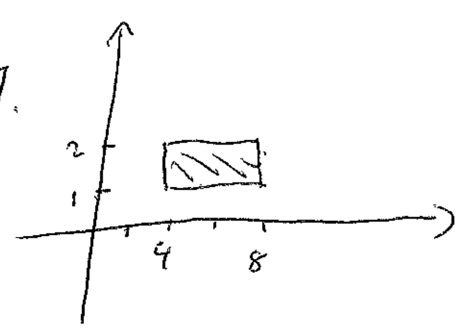
But this only gives $(0,0)$, so we ignore it.

$y=2x$: $\frac{uv}{v+1} = \frac{2u}{v+1} \rightsquigarrow v=2$ (or $u=0$)

$x+y=4$: $\frac{u+uv}{v+1} = 4 \rightsquigarrow u=4$

$x+y=8$: $\frac{u+uv}{v+1} = 8 \rightsquigarrow u=8$

We get the rectangle $[4,8]\times[1,2]$.



(b) $\text{Jac}(G) = \begin{vmatrix} \frac{1}{v+1} & \frac{-u}{(v+1)^2} \\ \frac{v}{v+1} & \frac{u(v+1) - uv}{(v+1)^2} \end{vmatrix}$

$$= \frac{1}{v+1}\left(\frac{u}{(v+1)^2}\right) - \frac{-uv}{(v+1)^3}$$

$$= \frac{u+uv}{(v+1)^3}$$

$$= \frac{u(v+1)}{(v+1)^3}$$

$$= \frac{u}{(v+1)^2}$$

$$\iint_D (x+y)\,dx\,dy = \int_1^2 \int_4^8 \frac{u\cancel{(v+1)}}{\cancel{v+1}} \cdot \frac{u}{(v+1)^2}\,du\,dv$$

$$= \int_1^2 \frac{1}{(v+1)^2}\,dv \cdot \int_4^8 u^2\,du$$

$$= \frac{-1}{v+1}\Big|_1^2 \cdot \frac{1}{3}u^3 \Big|_4^8$$

$$= \left[\frac{-1}{3} + \frac{1}{2}\right] \cdot \left[\frac{1}{3}(512 - 64)\right]$$

$$= \frac{1}{18}(448)$$

$$= \frac{224}{9}$$

Problem 3.

Compute the following.

(a) [5pts.] The total charge on a wire in the shape of the helix $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \leq t \leq 2\pi$ with charge density $f(x, y, z) = x + 2y + z$.

(b) [5pts.] The work done by a force field $\mathbf{F}(x, y, z) = \langle \frac{y}{1+x^2}, \arctan(x), 2z \rangle$ in moving a particle from $(0, 7, 1)$ to $(1, 8, e)$ along the path $\mathbf{r}(t) = \langle t^2, t+7, e^t \rangle$.

(a) $f(\vec{r}(t)) = \cos t + 2\sin t + t \qquad \vec{r}'(t) = \langle -\sin t, \cos t, 1 \rangle \qquad \|\vec{r}'(t)\| = \sqrt{2}$

$$\text{Total charge} = \int_0^{2\pi} f(\vec{r}(t)) \, \|\vec{r}'(t)\| \, dt$$

$$= \int_0^{2\pi} \sqrt{2}(\cos t + 2\sin t + t)\,dt$$

$$= \sqrt{2}\left[\sin t - 2\cos t + \tfrac{1}{2}t^2\right]_0^{2\pi}$$

$$= \sqrt{2}\left(\tfrac{1}{2}(2\pi)^2\right)$$

$$= 2\sqrt{2}\pi^2$$

(b) Let $f = y\arctan(x) + z^2$, so that $\vec{F} = \nabla f$. Then

$$\int_C \vec{F} \cdot d\vec{r} = f(1, 8, e) - f(0, 7, 1)$$

$$= (8\arctan(1) + e^2) - (7\arctan(0) + 1^2)$$

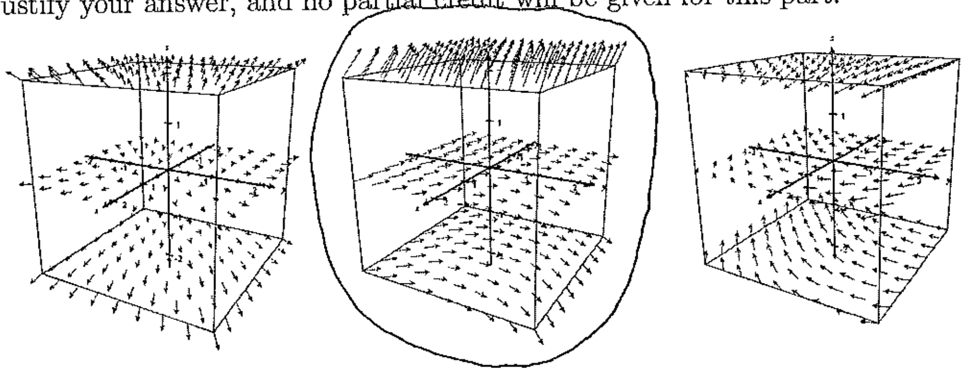
$$= 8\left(\tfrac{\pi}{4}\right) + e^2 - 1$$

$$= 2\pi + e^2 - 1$$

Problem 4.

Consider the vector field $\mathbf{F}(x,y,z) = \langle 2y, 4, e^z \rangle$.

(a) [3pts.] Which of the following is a picture of $\mathbf{F}$? Circle one; you do not need to justify your answer, and no partial credit will be given for this part.



(b) [3pts.] Compute $\operatorname{div}(\mathbf{F})$.

(c) [4pts.] Find the integral of $\mathbf{F}$ over the unit circle in the xy-plane oriented clockwise.

(b) $\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(2y) + \frac{\partial}{\partial y}(4) + \frac{\partial}{\partial z}(e^z) = e^z$

(c) $\vec{r}(t) = \langle \sin t, \cos t, 0 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^{2\pi} \langle 2\cos t, 4, 1 \rangle \cdot \langle \cos t, -\sin t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} (2\cos^2 t - 4\sin t)\,dt$$

$$= \int_0^{2\pi} (1 + \cos(2t) - 4\sin t)\,dt$$

$$= t + \tfrac{1}{2}\sin(2t) + 4\cos t \Big|_0^{2\pi}$$

$$= 2\pi$$

Problem 5.

Consider the vector field

$$\mathbf{F}(x,y) = \langle F_1, F_2 \rangle = \left\langle \frac{y}{x^2-y^2}, \frac{-x}{x^2-y^2} \right\rangle$$

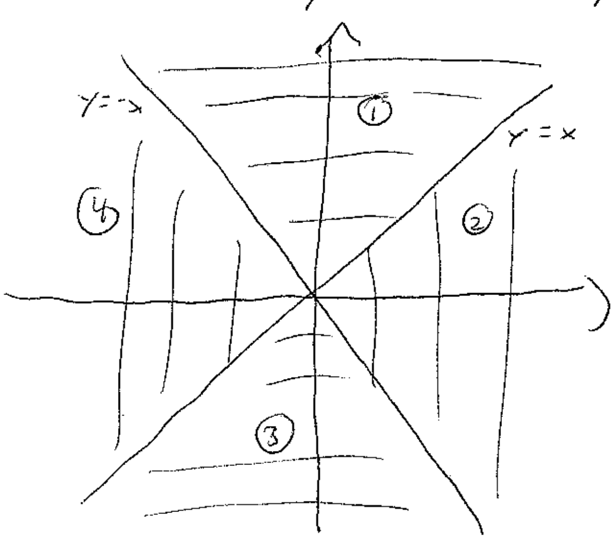
(a) [5pts.] Show that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

(b) [5pts.] Show that $\mathbf{F}$ is defined on four distinct connected domains in the plane. On each of these domains, is $\mathbf{F}$ conservative? (Hint: Are these domains simply connected?)

(a) $$\frac{\partial F_1}{\partial y} = \frac{1(x^2-y^2)+y(+2y)}{(x^2-y^2)^2} = \frac{x^2+y^2}{(x^2-y^2)^2}$$

$$\frac{\partial F_2}{\partial x} = \frac{-1(x^2-y^2)+(+x)(2x)}{(x^2-y^2)^2} = \frac{x^2+y^2}{(x^2-y^2)^2}$$

(b) $\vec{F}$ is defined everywhere but when $x^2-y^2=0$, or on the lines $x=y$ and $x=-y$.



This leaves four connected, indeed simply-connected, domains on the plane. Since $\vec{F}$ satisfies the cross-partials condition on each of the domains, $\vec{F}$ must be conservative on each one.