Math 32B, Lecture 4
Multivariable Calculus

Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.

Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Name: Solutions

UID: __________

Section: __________

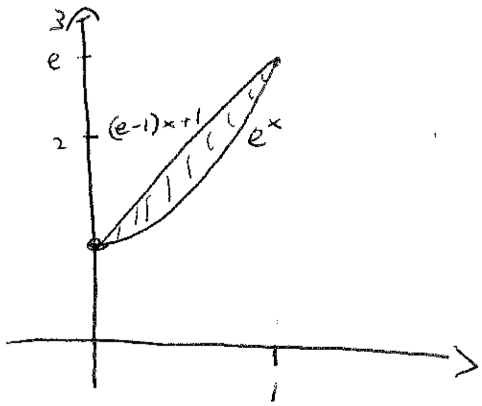
Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

(a) [5pts.] Draw the domain in the plane bounded by $y = e^x$ and $y = ex - x + 1$.

(b) [5pts.] Integrate the function $f(x, y) = x$ over the domain from part (a).

(a)



(b) $\int_0^1 \int_{e^x}^{(e-1)x+1} x\,dy\,dx = \int_0^1 \left((e-1)x^2 + x - xe^x \right) dx$

$$\begin{array}{ccc} x & & e^x \\ & + & \\ 1 & & e^x \\ & - & \\ 0 & & e^x \end{array}$$

$= \left[\frac{1}{3}(e-1)x^3 + \frac{1}{2}x^2 - (xe^x - e^x)\right]_0^1$

$= \left[\frac{1}{3}(e-1) + \frac{1}{2} - e + e\right] - [0 + 0 - 0 + 1]$

$= \frac{e}{3} - \frac{1}{3} + \frac{1}{2} - 1$

$= \frac{e}{3} + \frac{-5}{6}$

$= \frac{2e-5}{6}$

Problem 2.

(a) [5pts.] Draw the polar curve $r = \cos\theta + \sin\theta$. (Hint: It may be extremely helpful to multiply this equation by r.)

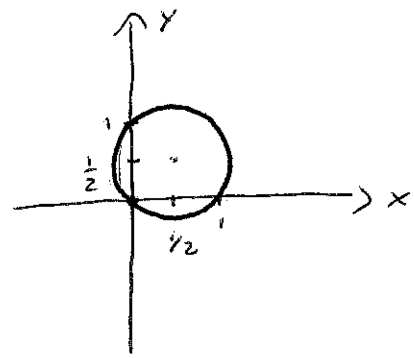
(b) [5pts.] Find the area of the domain in the first quadrant bounded by this curve and the axes.

(a) $r^2 = r\cos\theta + r\sin\theta$

$x^2 + y^2 = x + y$

$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) = \frac{1}{2}$

$\left.\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{2}\ \right\}$ circle!



(b) $$\int_0^{\pi/2}\int_0^{\cos\theta+\sin\theta} r\,dr\,d\theta = \int_0^{\pi/2} \frac{1}{2}r^2\Big|_0^{\cos\theta+\sin\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\left(\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\left(1 + \sin(2\theta)\right) d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{1}{2}\cos(2\theta)\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\right) - \left(0 - \frac{1}{2}(1)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 1\right]$$

$$= \frac{\pi + 2}{4}$$

Problem 3.

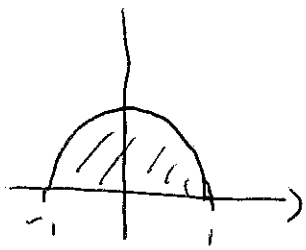
Evaluate the integrals. Use any method you like.

(a) [5pts.] $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{x^2+y^2}} 2\,dz\,dy\,dx$

(b) [5pts.] $\int_{0}^{1}\int_{x}^{1} xe^{y^3}\,dy\,dx$

ⓐ Change to cylindrical

xy-region



$$\int_0^{\pi}\int_0^1\int_0^r 2r\,dz\,dr\,d\theta$$

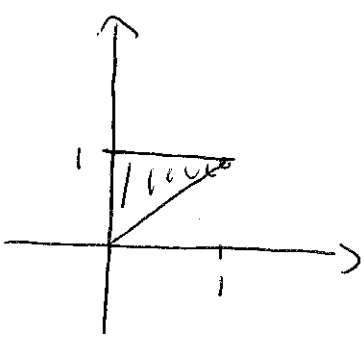
$$= \int_0^{\pi}\int_0^1 2r^2\,dr\,d\theta$$

$$= \left(\int_0^{\pi} d\theta\right)\cdot\left(\int_0^1 2r^2\,dr\right)$$

$$= \pi \cdot \tfrac{2}{3}r^3\Big|_0^1$$

$$= \tfrac{2}{3}\pi$$

ⓑ Change order



$$\int_0^1\int_0^y xe^{y^3}\,dx\,dy = \int_0^1 \tfrac{1}{2}y^2e^{y^3}\,dy$$

$$= \tfrac{1}{6}e^{y^3}\Big|_0^1$$

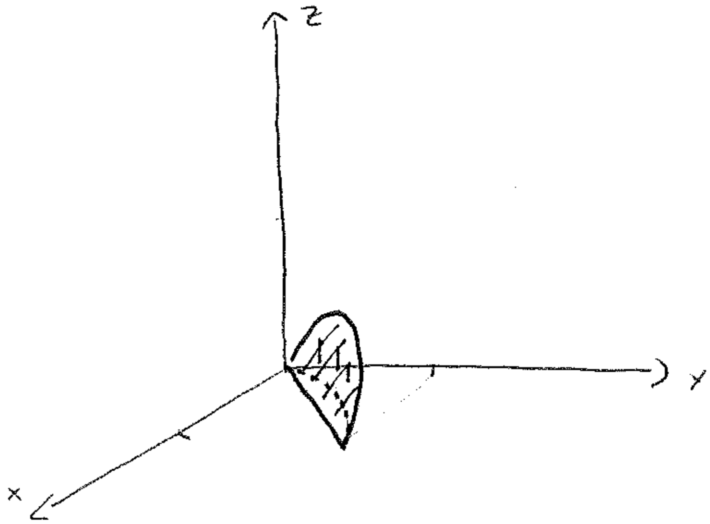
$$= \tfrac{1}{6}\left[e-1\right]$$

Problem 4.

(a) [5pts.] Draw the solid in $\mathbb{R}^3$ bounded by $y = \sqrt{x}$, $y = x$, and $z = y - x$.

(b) [5pts.] Write down three different triple integrals that compute the volume of the solid you drew in part (a). You do not need to evaluate them.

(a)



(b)

$$\int_0^1 \int_x^{\sqrt{x}} \int_0^{y-z} dz\,dy\,dx$$

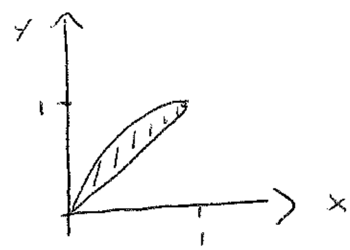
$$\int_0^1 \int_{y^2}^{y} \int_0^{y-x} dz\,dx\,dy$$

$$\int_0^1 \int_0^{y(1-y)} \int_{y^2}^{y-z} dx\,dz\,dy$$

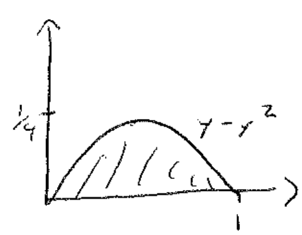
$$\int_0^1 \int_0^{\sqrt{x}-x} \int_{z+x}^{\sqrt{x}} dy\,dz\,dx$$

etc.

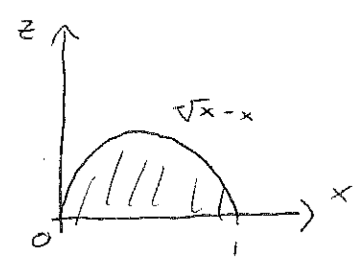
xy - projection



yz - projection



xz - projection

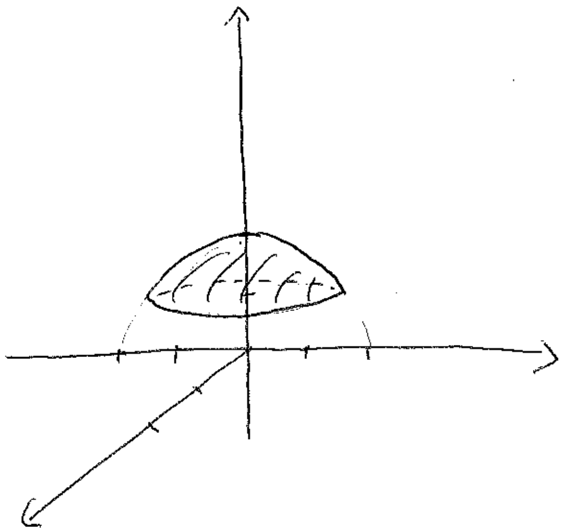


Problem 5.

(a) [5pts.] Draw the region in $\mathbb{R}^3$ bounded by the surfaces $z = 1$ and $x^2 + y^2 + z^2 = 4$.

(b) [5pts.] Using any method you like, compute the integral of $f(x, y, z) = z$ over the region in part (a).

(a)



(b) Spherical Coordinates

$z = 1$ $\quad\quad x^2+y^2+z^2=4$

$\rho\cos\phi = 1$ $\quad\quad \rho = 2$

$\rho = \frac{1}{\cos\phi}$

Intersect at

$\frac{1}{\cos\phi} = 2$

$\frac{1}{2} = \cos\phi$

$\frac{\pi}{3} = \phi$

$$\int_0^{2\pi}\int_0^{\pi/3}\int_{\frac{1}{\cos\phi}}^{2} (\rho\cos\phi)\,\rho^2\sin\phi \; d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi/3} \frac{1}{4}\rho^4 \Big|_{\frac{1}{\cos\phi}}^{2} \cos\phi\sin\phi \, d\phi$$

$$= 2\pi \int_0^{\pi/3} \left[4\cos\phi\sin\phi - \frac{1}{4}\frac{\sin\phi}{\cos^3\phi}\right] d\phi$$

$$= 2\pi \left[-4 \cdot \frac{1}{2} \cos^2 \phi + \frac{1}{4} \cdot \frac{1}{2} \cos^{-2} \phi \right]_0^{\pi/3}$$

$$= 2\pi \left[\left(-2 \left(\frac{1}{2} \right)^2 + \frac{1}{8} \left(\frac{1}{2} \right)^{-2} \right) - \left(-2 + \frac{1}{8} \right) \right]$$

$$= 2\pi \left[-\frac{1}{2} + \frac{1}{2} - \left(-\frac{15}{8} \right) \right]$$

$$= \frac{15\pi}{4}$$