Math 32B Midterm 1R

MELODY CHEN

TOTAL POINTS

39 / 40

QUESTION 1

1 Product rule 5 / 5

✓ **+ 2 pts** Correct expression for curl operator in components

✓ **+ 1 pts** Correct expression for $$\nabla f$$

✓ **+ 1 pts** Correct application of the product rule for partial derivatives

✓ **+ 1 pts** Solution clearly explained

+ 0 pts No credit due

QUESTION 2

2 Line integral 8 / 8

✓ **+ 4 pts** Parametrized curve correctly, except possibly orientation

✓ **+ 2 pts** Correct orientation of curve/integral

✓ **+ 2 pts** Successful computation of the correct integral, except possibly orientation/sign

✓ **+ 1 pts** Clearly explaining your solution

+ 1 pts Bonus: sketch of curve (with or without orientation)

+ 0 pts No points

QUESTION 3

3 Fundamental Theorem of Vector Line Integrals 12 / 12

✓ **+ 1 pts** correct answer (-sqrt(2) - pi/2), correctly derived using potential function, and solution is clearly explained

✓ **+ 9 pts** correct potential function f(x,y,z) = sqrt(1+y^2) + cos(x-z) - z

✓ **+ 2 pts** integral is equal to f(pi,0,pi/2) - f(0,1,0)

+ 1 pts (incorrect) integral is equal to f(pi,0,pi/2) - f(0,1,0)

+ 0 pts no points

+ 7 pts partial credit for nearly correct expression for potential function

+ 3 pts partial credit for some progress towards finding a potential function

+ 2 pts correct expression for a parametric curve from (0,1,0) to (pi,0,pi/2) (only if no solution via potential function)

+ 2 pts correct expression for a vector line integral using a parametric curve (only if no solution via potential function)

+ 8 pts correct answer (-sqrt(2) - pi/2), correctly derived via a parametric curve, and solution is clearly explained

+ 1 pts partial credit for incorrect expression for parametric curve (only if no solution via potential function)

QUESTION 4

4 Volume via a double integral 14 / 15

✓ **+ 1 pts** Clearly written attempt

✓ **+ 2 pts** Drawing and labelling of region (4 pts)

✓ **+ 1 pts** Drawing and labelling of region

✓ **+ 1 pts** Draw/label region (deriving x-intersection points)

✓ **+ 2 pts** Correctly set-up integral (5 pts)

✓ **+ 2 pts** Correctly set-up integral

✓ **+ 1 pts** Correctly set-up integral

✓ **+ 2 pts** Showing z = 2 - x^2 is the top surface (2 pts)

✓ **+ 2 pts** Computation (3 pts max)

✓ **+ 1 pts** Computation

- 1 Point adjustment

Need to algebraically show why 2 - x^2 > x^2, can't just plug in two endpoints

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 1
2/1/2019

Time Limit: 50 Minutes

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3C Version (→)

This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

1. (5 points) Let $f(x, y, z)$ be a smooth scalar field and $\mathbf{F}(x, y, z)$ be a smooth vector field. Show that

$$\operatorname{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f \operatorname{curl} \mathbf{F}.$$

$$\operatorname{curl}(f\vec{F}) = \nabla \times f\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix}$$

$$= \left\langle \frac{\partial fF_3}{\partial y} - \frac{\partial fF_2}{\partial z}, \frac{\partial fF_1}{\partial z} - \frac{\partial fF_3}{\partial x}, \frac{\partial fF_2}{\partial x} - \frac{\partial fF_1}{\partial y} \right\rangle$$

use product rule

$$= \left\langle f\frac{\partial F_3}{\partial y} + F_3\frac{\partial f}{\partial y} - \left(f\frac{\partial F_2}{\partial z} + F_2\frac{\partial f}{\partial z}\right), f\frac{\partial F_1}{\partial z} + F_1\frac{\partial f}{\partial z} - f\frac{\partial F_3}{\partial x} - F_3\frac{\partial f}{\partial x}, f\frac{\partial F_2}{\partial x} + F_2\frac{\partial f}{\partial x} - f\frac{\partial F_1}{\partial y} - F_1\frac{\partial f}{\partial y} \right\rangle$$

$$= f\left\langle \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \right\rangle + \left\langle F_3\frac{\partial f}{\partial y} - F_2\frac{\partial f}{\partial z}, F_1\frac{\partial f}{\partial z} - F_3\frac{\partial f}{\partial x}, F_2\frac{\partial f}{\partial x} - F_1\frac{\partial f}{\partial y} \right\rangle$$

$$= f \operatorname{curl}\vec{F} + \nabla f \times \vec{F}$$

same

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

same

$$\nabla f \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left\langle F_3\frac{\partial f}{\partial y} - F_2\frac{\partial f}{\partial z}, F_1\frac{\partial f}{\partial z} - F_3\frac{\partial f}{\partial x}, F_2\frac{\partial f}{\partial x} - F_1\frac{\partial f}{\partial y} \right\rangle$$

The above computations show the proof of $\operatorname{curl}(f\vec{F}) = \nabla f \times \vec{F} + f \operatorname{curl}\vec{F}$.

2. (8 points) Let $C \subset \mathbb{R}^2$ be the part of the curve $y = x^4$ from $x = 1$ to $x = -1$. Find

$$\int_C (1+x)\,dy - y\,dx.$$

$y = x^4$ $x = t$

$y = t^4$

$\vec{F} = \langle -y, 1+x \rangle$

$\vec{r}(t) = \langle t, t^4 \rangle$

$\int_{-1}^{1} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$ $\vec{r}\,'(t) = \langle 1, 4t^3 \rangle$

$= \int_{-1}^{1} \langle -t^4, 1+t \rangle \cdot \langle 1, 4t^3 \rangle\,dt$ $-1 \le t \le 1$

$= \int_{-1}^{1} (-t^4 + 4t^3 + 4t^4)\,dt$

$= \left[\frac{3}{5}t^5 + t^4 \right]_{-1}^{1}$

$= \left(\frac{3}{5} + 1\right) - \left(-\frac{3}{5} + 1\right)$

$= \frac{3}{5} + \frac{3}{5} = \frac{6}{5}$

Since the curve goes from t=1 to t=-1, the integral bounds should be flipped, so answer is negative

$\int_C (1+x)dy - y\,dx$ for curve $y = x^4$ from $x=1$ to $x=-1$ is

$$\boxed{-\frac{6}{5}}$$

3. (12 points) Let

$$\mathbf{F}(x,y,z) = \left\langle -\sin(x-z)\,,\ \frac{y}{\sqrt{1+y^2}}\,,\ \sin(x-z)-1 \right\rangle.$$

Find $\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ where C is any smooth curve from $(0,1,0)$ to $(\pi,0,\frac{\pi}{2})$.

Find a potential function $\vec{F}=\nabla f$

$\frac{\partial f}{\partial x} = -\sin(x-z)$ $\quad f=\int -\sin(x-z)\,dx$

$= \cos(x-z) + g(y,z)$.

$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{1+y^2}} = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}$.

$g = \int \frac{y}{\sqrt{1+y^2}}\,dy = \sqrt{1+y^2} + h(z)$ $\quad u = 1+y^2,\ du = 2y\,dy$

$\frac{\partial f}{\partial z} = \sin(x-z) - 1$ $\quad \frac{\partial(\cos(x-z)+\sqrt{1+y^2}+h(z))}{\partial z}$

~~$\sin(x-z)$~~ -1 = ~~$+\sin(x-z)$~~ $+ h'(z)$.

$h'(z) = -1$ $\quad h(z) = -z$

This is the potential function $\to$ $f = \cos(x-z) + \sqrt{1+y^2} - z$ $\quad$ assume constant is zero.

using FToVLI

$\int_C F(x,y,z)\,dr$ from $(0,1,0)$ to $(\pi,0,\frac{\pi}{2})$

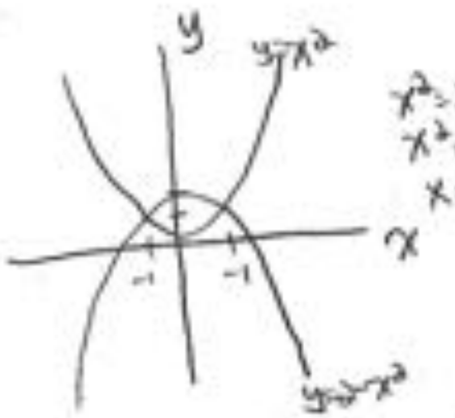
$= f(\pi,0,\frac{\pi}{2}) - f(0,1,0)$

$= \left(\cos(\pi-\frac{\pi}{2}) + \sqrt{1} - \frac{\pi}{2}\right) - (1+\sqrt{2})$

$= 1 - \frac{\pi}{2} - 1 - \sqrt{2} = \boxed{-\frac{\pi}{2}-\sqrt{2}}$

The answer for $\int_C \vec{F}(x,y,z)\,d\vec{r}$ from $(0,1,0)$ to $(\pi,0,\frac{\pi}{2})$ is $-\frac{\pi}{2}-\sqrt{2}$.

4. (15 points) Find the volume of the region $\mathcal{W}$ bounded by the surfaces $y = x^2$, $y = 2 - x^2$, $z = x^2$, $z = 2 - x^2$.



$x^2 = 2 - x^2$ D: $\{-1 \le x \le 1,\ x^2 \le y \le 2 - x^2\}$.
$x^2 = 1$
$x = \pm 1$

when $x = 0$ ... $2 - x^2 > x^2$
when $x = 0.5$ $2 - x^2 > x^2$

Thus, $h = 2 - x^2 - x^2 = 2 - 2x^2$.

$$V = \int_{-1}^{1}\int_{x^2}^{2-x^2} 2 - 2x^2 \, dy\, dx$$

$$= \int_{-1}^{1} \left[2y - 2yx^2\right]_{x^2}^{2-x^2} dx \qquad \begin{aligned} &2x^2(2-x^2) \\ &= 4x^2 - 2x^4 \end{aligned}$$

$$= \int_{-1}^{1} \left[2(2-x^2) - 2(2-x^2)x^2 - 2x^2 + 2x^4\right] dx$$

$$= \int_{-1}^{1} (4 - 2x^2 - 4x^2 + 2x^4 - 2x^2 + 2x^4)\, dx$$

symmetric can change bound $\leftarrow$ $= \int_{-1}^{1} (4 - 8x^2 + 4x^4)\, dx$

$$= 2 \left[4x - \frac{8}{3}x^3 + \frac{4}{5}x^5\right]_0^1$$

$$= 2\left(4 - \frac{8}{3} + \frac{4}{5}\right) = 8 - \frac{16}{3} + \frac{8}{5}$$

The volume of region W bounded by the surface is $\underline{8 - \frac{16}{3} + \frac{8}{5}}$.