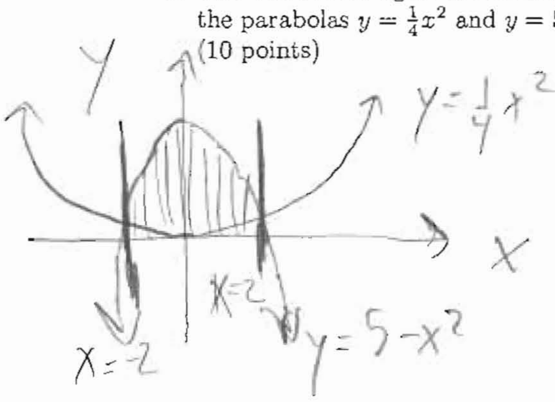
1. Use a double integral to find the volume under the graph of $f(x, y) = 4$ over the region bounded by the parabolas $y = \frac{1}{4}x^2$ and $y = 5 - x^2$. Include a sketch of the region of integration in the xy-plane. (10 points)



$5 - x^2 = \frac{1}{4}x^2$

$5 = \frac{5}{4}x^2$

$4 = x^2$

$x = \pm 2$

$$\iint_D f(x,y)\,dA = \int_{-2}^{2}\int_{\frac{x^2}{4}}^{5-x^2} 4\,dy\,dx = 8\int_0^2\int_{\frac{x^2}{4}}^{5-x^2} dy\,dx$$

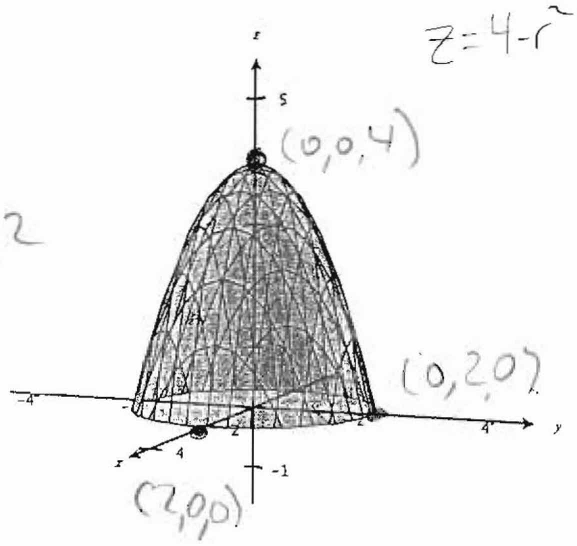
$$= 8\int_0^2\left[5 - x^2 - \frac{x^2}{4}\right]dx = 8\int_0^2\left[5 - \frac{5x^2}{4}\right]dx = 40\left[x - \frac{x^3}{12}\right]_0^2$$

$$40\left[2 - \frac{\cancel{8}^{\,2}}{\cancel{12}_{\,3}}\right] = 40\left[\frac{6}{3} - \frac{2}{3}\right] = \frac{40 \cdot 4}{3} = \boxed{\frac{160}{3}}$$

2. The surface shown below is a paraboloid with circular xy-cross-sections. It passes through the points $(2,0,0)$, $(0,2,0)$, and $(0,0,4)$. Let E be the solid contained between the paraboloid and the xy-plane.

$z = 4 - r^2$

$z = 4 - x^2 - y^2$



$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4-x^2}$

(a) Set up (but do not evaluate) an iterated double integral in rectangular coordinates to compute the volume of E. Make sure to include all limits of integration and the function to be integrated. (6 points)

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - y^2)\,dy\,dx$$

✓

(b) Set up (but do not evaluate) an iterated triple integral in rectangular coordinates to compute the volume of E. Make sure to include all limits of integration and the function to be integrated. (6 points)

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{(4-x^2-y^2)} 1 \cdot dz\,dy\,dx$$

✓

$$\text{Avg} = \frac{\int_D f(x,y)\,dA}{\text{Area}}$$

3. (a) Compute the average value of the function $f(x, y) = xy$ on the box $[-1, 1] \times [-1, 1]$. (5 points)

$$\int_{-1}^{1}\int_{-1}^{1} xy\,dx\,dy = \int_{-1}^{1}\left[\frac{x^2y}{2}\right]_{-1}^{1} dy = \int_{-1}^{1}\left[\frac{y}{2}-\frac{y}{2}\right]dy$$

$$= \int_{-1}^{1} 0 \cdot dy = 0$$

$$\frac{0}{(1--1)(1--1)} = \frac{0}{2\cdot 2} = \boxed{0}$$

(b) Would the average value of $f(x, y) = xy$ be higher or lower on the box $[0, 2] \times [0, 2]$? Justify your answer in words or with a computation. (3 points)

Higher

$$\int_{0}^{2}\int_{0}^{2} xy\,dx\,dy = \int_{0}^{2}\left[\frac{x^2y}{2}\right]_{0}^{2} dy = \int_{0}^{2}\left(\frac{4y}{2}\right)dy$$

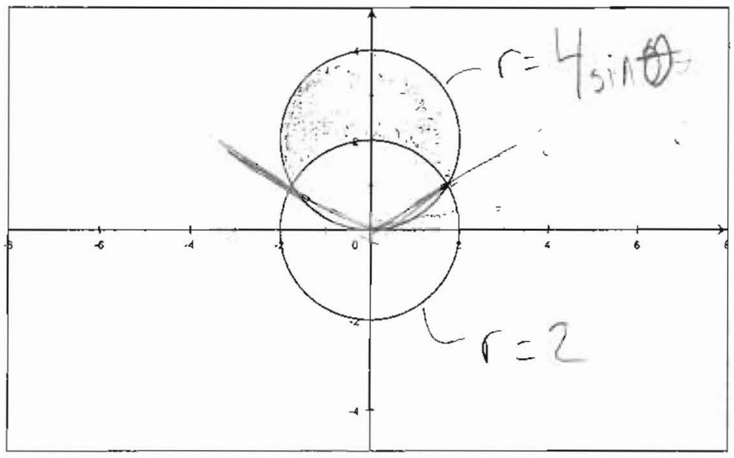
$$= \left[\frac{2y^2}{2}\right]_{0}^{2} = 4$$

$$\frac{4}{(2-0)(2-0)} = 1 = \text{avg value}$$

4. Both curves shown below are circles. Set up (but do not evaluate) an integral *in polar coordinates* to compute the shaded area between the circles. (8 points)

$\theta = \frac{\pi}{2}$

$\theta = \pi$



$r = 4\sin\theta$

$r = 2$

$\theta = 0$

$\theta = \frac{3\pi}{2}$

$\cancel{2+2\sin\theta = 2}$

$\cancel{2\sin\theta = 0}$

$\cancel{\sin\theta = 0}$

$\cancel{\theta = 0}$

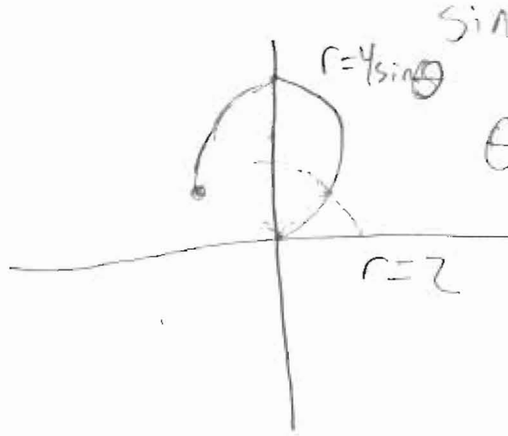
$2 + 2 \cdot \frac{\sqrt{2}}{2} \quad 2+\sqrt{2}$

$\cancel{\int_{\pi/4}^{3\pi/4} \int_{2}^{2+2\sin\theta} r\,dr\,d\theta}$

$4\sin\theta = 2$

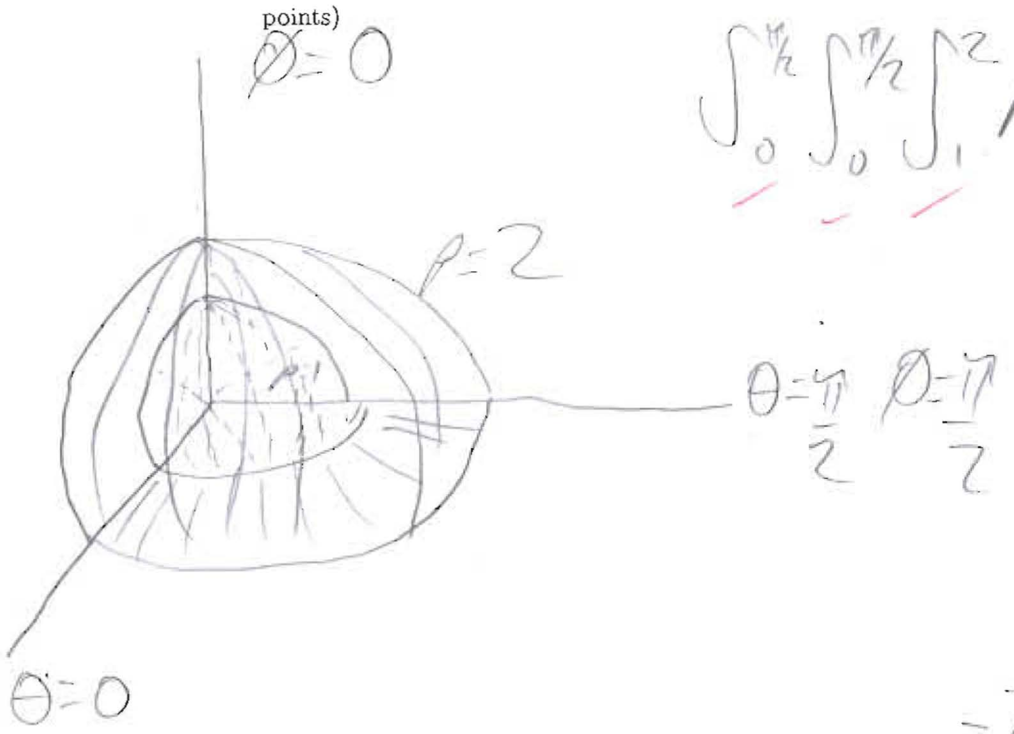
$\sin\theta = \frac{1}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$



$$\boxed{\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{2}^{4\sin\theta} r\,dr\,d\theta}$$ ✓

5. Use a triple integral *not in rectangular coordinates* to compute the volume in the first octant between the sphere of radius 1 centered at the origin and the sphere of radius 2 centered at the origin. (12 points)



$$\int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\frac{\pi}{2}\cdot\int_0^{\pi/2}\left[\frac{\rho^3}{3}\right]_1^2 \sin\phi \, d\phi$$

$$\frac{\pi}{2}\int_0^{\pi/2}\left[\frac{8}{3}-\frac{1}{3}\right]\sin\phi \, d\theta$$

$$\frac{\pi}{2}\int_0^{\pi/2}\frac{7}{3}\sin\phi \, d\theta$$

$$-\frac{7\pi}{6}\left[\cos\phi\right]_0^{\pi/2}$$

$$-\frac{7\pi}{6}\left[0-1\right]$$

$$\boxed{\frac{7\pi}{6}}$$

12

$\frac{4}{3}\pi r^3$

$\frac{4\pi}{3}$ $\quad \frac{4}{3}\pi\cdot 8 = \frac{32\pi}{3} \div 8 = \frac{4\pi}{3}$

$\div 8 = \frac{\pi}{6}$

$\frac{4\pi}{3} - \frac{\pi}{6}$

$\frac{8\pi}{6} - \frac{\pi}{6}$

$\frac{7\pi}{6}$