Midterm 2
Multivariable Calculus
(Math 32B-001)

Show your work to receive partial credits. Use of calculator is NOT allowed for this exam.

Name:

TA's Name: Ha TA Meeting Day: Thurs

Question:	1	2	3	4	Total
Points:	5	5	5	5	20
Score:	5	5	5	4	19

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$\boxed{\text{5 points}}$ Use the Change of Variables Formula $G(u,v) = (u/v, uv)$ to evaluate the following integral

(uv^{-1}, uv)

$$\int\int_{\mathcal{D}} (x^2+y^2)\,dx\,dy,$$

where $\mathcal{D}$ is the domain $1 \le xy \le 4, 1 \le y/x \le 4$.

$$\text{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = (v^{-1})(u) - (-uv^{-2})(v)$$

$$= \frac{u}{v} - \left(-\frac{u}{v^2}\cdot v\right) = \frac{u}{v} + \frac{u}{v} = 2uv^{-1}$$

bounds:

$1 = xy \to 1 = \left(\frac{u}{v}\cdot uv\right)$ $\quad u = \pm 1$

$4 = xy \to$ $\quad u = \pm 2$ (take positive) so it's one to one

$1 = \frac{y}{x} \to 1 = \frac{uv\cdot v}{\cancel{u}} \to$ $\quad v = \pm 1$

$4 = \frac{y}{x} \to$ $\quad v = \pm 2$

$$x^2+y^2 = \frac{u^2}{v^2} + u^2v^2$$

$$\int_1^2\int_1^2 2uv^{-1}(u^2v^{-2} + u^2v^2)\,du\,dv = 2\int_1^2\int_1^2 (u^3v^{-3} + u^3v)\,du\,dv$$

$$= 2\int_1^2\int_1^2 u^3(v^{-3}+v)\,du\,dv = 2\left[\frac{u^4}{4}\right]_1^2 \cdot \left[-\frac{v^{-2}}{2} + \frac{v^2}{2}\right]_1^2$$

$$= 2\left(4 - \frac{1}{4}\right)\left(-\frac{1}{8} + 2 + \cancel{\frac{1}{2}} - \frac{1}{2}\right)$$

$$= 2\left(\frac{15}{4}\right)\left(\frac{15}{8}\right) = \left(\frac{15}{4}\right)^2 = \frac{225}{16}$$

2. $\boxed{\text{5 points}}$ Use spherical coordinates to evaluate the integral $\iiint_W f(x,y,z)\,dV$, where $f(x,y,z) = z$ and W is the ice cream shaped region lying above the cone $x^2+y^2=z^2$ and below the sphere $x^2+y^2+z^2=4$.

$x^2+y^2=z^2$, $x^2+y^2+z^2=4$

$2z^2 = 4$

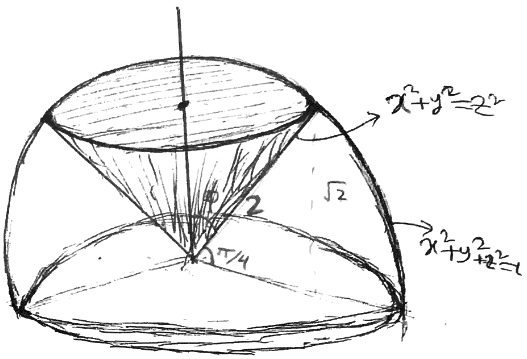
$z = \sqrt{2}$

intersect at $z=\sqrt{2}$

$0 \le \phi \le \frac{\pi}{4}$

$0 \le \theta \le 2\pi$

$0 \le \rho \le 2$



$\cancel{x^2+y^2=z^2} \rightarrow \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta = \rho^2\cos^2\phi$

$\rho^2\sin^2\phi$

$\cancel{\cos\phi\sin\phi = \frac{1}{2}\sin^2\phi}$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{2} \rho^3\cos\phi\sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}} \left[\frac{\rho^4}{4}\cos\phi\sin\phi\right]_0^2 d\phi\, d\theta$$

$$= 4\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \cancel{4}\cos\phi\sin\phi \; d\phi\, d\theta$$

let $u = \sin\phi$
$du = \cos\phi \, d\phi$
$\int u\,du = \frac{1}{2}u^2$

$\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

$$= 4\int_0^{2\pi}\left[\frac{\sin^2\phi}{2}\right]_0^{\frac{\pi}{4}} d\theta = 4\int_0^{2\pi}\frac{\left(\frac{1}{\sqrt{2}}\right)^2}{2}\,d\theta = 4\left[\frac{\theta}{4}\right]_0^{2\pi} = 2\pi$$

3. $\boxed{\text{5 points}}$ Find the center of mass of the region $\mathcal{D}$ bounded by $y^2 = x + 4$ and $x = 0$ with mass density $\delta(x, y) = |y|$.

$x = y^2 - 4$

$y = \pm\sqrt{x+4}$

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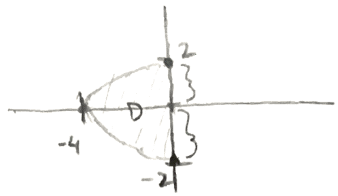

$$\delta(x,y) = |y| \rightarrow \begin{cases} y, & y > 0 \\ -y, & y < 0 \end{cases}$$

$$M = \int_{-4}^{0}\int_{0}^{\sqrt{x+4}} y\,dy\,dx + \int_{-4}^{0}\int_{-\sqrt{x+4}}^{0} -y\,dy\,dx$$

$$= \int_{-4}^{0}\left[\frac{y^2}{2}\right]_0^{\sqrt{x+4}} dx + \int_{-4}^{0}\left[-\frac{y^2}{2}\right]_{-\sqrt{x+4}}^{0} dx$$

$$= \int_{-4}^{0} \frac{x}{2} + 2\,dx + \int_{-4}^{0}\left(\frac{x}{2}+2\right)dx = 2\left[\frac{x^2}{4} + 2x\right]_{-4}^{0} = 2(-4+8) = 8$$

symmetry

$$M_y = 2\int_{-4}^{0}\int_{0}^{\sqrt{x+4}} xy\,dy\,dx = 2\int_{-4}^{0}\left[x\frac{y^2}{2}\right]_0^{\sqrt{x+4}} dx = 2\int_{-4}^{0}\left(\frac{x^2}{2}+2x\right)dx = 2\left[\frac{x^3}{6}+x^2\right]_{-4}^{0}$$

$$= 2\left(\frac{64}{6} - 16\right) = \frac{64}{3} - 32 = -\frac{32}{3}$$

$$x_{cm} = \frac{\left(-\frac{32}{3}\right)}{8} = -\frac{32}{3}\cdot\frac{1}{8} = -\frac{4}{3}$$

By symmetry (or by this calculation), $y_{cm} = 0$

$$M_x = 2\int_{-4}^{0}\int_{0}^{\sqrt{x+4}} y^2\,dy\,dx = 2\int_{-4}^{0}\left[\frac{y^3}{3}\right]_0^{(x+4)^{1/2}} dx = \frac{2}{3}\int_{-4}^{0}(x+4)^{3/2}\,dx$$

$u = x+4$, $du = dx$, $\int u^{3/2} = \frac{2}{5}u^{5/2}$

$$= \frac{4}{15}\left[(x+4)^{5/2}\right]_{-4}^{0} = 0$$

Therefore the center of mass is at the point $\left(-\frac{4}{3}, 0\right)$

of the region D bounded by $y^2 = x+4$ and $x = 0$

$x = y^2 - 4$

$y = \pm\sqrt{x+4}$

4. $\boxed{\text{5 points}}$ Evaluate $\oint_C (\sin x\, dx + z\cos y\, dy + \sin y\, dz)$, where C is the ellipse $4x^2+9y^2 = 36$, oriented counterclockwise.

on the x-y plane

$\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \to 0 = 0$

$\vec{F} = \langle \sin x,\ z\cos y,\ \sin y \rangle$

Try to find a potential function

$\frac{\partial V}{\partial x} = \sin x, \quad V = -\cos x + C_1$

$\frac{\partial V}{\partial y} = z\cos y, \quad V = z\sin y + C_2$

$\frac{\partial V}{\partial z} = \sin y, \quad V = z\sin y + C_3$

we see that there is a potential function $V = z\sin y - \cos x$.

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Since the domain over which the curve C is being integrated is simply connected, this indicates that the given field is a conservative vector field.
Therefore, evaluating the integral over any closed loop will result in 0.

$\oint_C (\sin x\, dx + z\cos y\, dy + \sin y\, dz) = 0$