Midterm 1
Multivariable Calculus
(Math 32B-001)

... to receive partial credits. Use of calculator is NOT allowed for this exam.

Na...

TA's Name: Ha TA Meeting Day: Thursday 10-11
Boelter 5422

Question:	1	2	3	4	Total
Points:	10	5	5	5	25
Score:	10	2+1 = 3	5	5	22+1 = 23

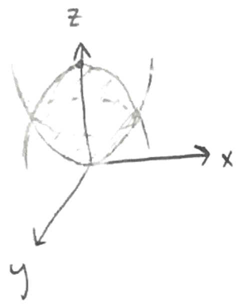
$\frac{256}{3}$ 768

$768 - 256 = 512$

1. (a) $\boxed{\text{5 points}}$ Sketch the region bounded by the two paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. Then set up a **double integral** which gives the volume of this region. (Do not evaluate the double integral).

(b) $\boxed{\text{5 points}}$ Compute the volume of the region bounded by $z = 16 - y$, $z = y$, $y = x^2$, and $y = 8 - x^2$

a)

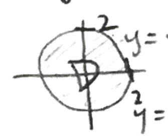


5

intersect

$x^2 + y^2 = 8 - x^2 - y^2$

$x^2 + y^2 = 4$



$y = \sqrt{4-x^2}$

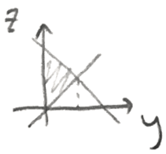
$y = -\sqrt{4-x^2}$ → from $z = x^2+y^2$ to $z = 8 - x^2 - y^2$

$8 - x^2 - y^2 - (x^2+y^2)$
$= 8 - 2x^2 - 2y^2$

volume of region:

$$\int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (8 - 2x^2 - 2y^2)\,dy\,dx$$

b)



5

$8 - x^2 = x^2$
$8 = 2x^2$
$x = \pm 2$

the volume of the region is given by

$$\int_{-2}^{2} \int_{x^2}^{8-x^2} (16 - 2y)\,dy\,dx$$

$$= \int_{-2}^{2} \left[16y - y^2\right]_{x^2}^{8-x^2} dx = \int_{-2}^{2} 16(8-x^2) - (8-x^2)^2 - (16x^2 - x^4)\,dx$$

$$= \int_{-2}^{2} (128 - 16x^2 - (x^4 - 16x^2 + 64) - 16x^2 + x^4)\,dx$$

$$= \int_{-2}^{2} (64 - 16x^2)\,dx = \left[64x - \frac{16}{3}x^3\right]_{-2}^{2} = 128 - \frac{128}{3} - \left(-128 + \frac{128}{3}\right)$$

$$= 256 - \frac{256}{3} = \frac{512}{3}$$

2. $\boxed{\text{5 points}}$ Prove the inequality $\displaystyle\iint_D \frac{2}{1+x^2+y^2}\,dA \le 8\pi$, where D is the disk $x^2+y^2 \le 4$.

For this D we can change to polar coordinates

let $u = 1+r^2$

$du = 2r\,dr$

$$\int_0^{2\pi}\int_0^{2} \frac{2}{1+r^2(\cancel{\sin^2\theta+\cos^2\theta})}\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} \frac{2r}{1+r^2}\,dr\,d\theta.$$

$\int u^{-1}du = \ln|u| = \ln(1+r^2)$

$$= \int_0^{2\pi} \Big[\ln(1+r^2)\Big]_0^2\, d\theta$$

$$= \int_0^{2\pi} \ln 5\, d\theta = \Big[(\ln 5)\theta\Big]_0^{2\pi} = \underline{2\pi \cdot \ln 5 \le 8\pi}$$

(because $\ln 5 \le 4$)

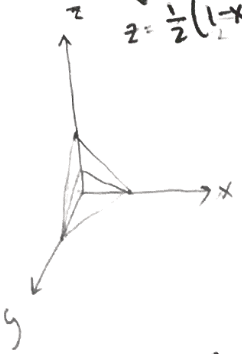
How do you know that?

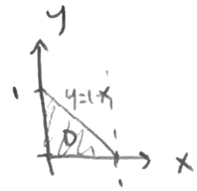
$2+1=3$

3. $\boxed{\text{5 points}}$ Using **triple integration** to compute the volume of the solid region W which lies in the first octant: $x \geq 0, y \geq 0, z \geq 0$ and bounded by the planes $x + y + z = 1$ and $x + y + 2z = 1$.

$z = \frac{1}{2}(1-x-y)$

$z = 1-x-y$





$$\iint_D \left(\int_{\frac{1}{2}(1-x-y)}^{1-x-y} 1 \, dz \right) dA = \int_0^1 \int_0^{1-x} 1-x-y-\tfrac{1}{2}(1-x-y) \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} (1-x-y) \, dy\, dx = \frac{1}{2} \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \frac{1}{2} \int_0^1 1-x-x(1-x)-\tfrac{1}{2}(1-x)^2 \, dx$$

$$= \frac{1}{2} \int_0^1 1-x-x+x^2-\tfrac{1}{2}(x^2-2x+1) \, dx$$

$$= \frac{1}{4} \int_0^1 (x^2-2x+1) \, dx = \frac{1}{4} \left[\frac{x^3}{3} - x^2 + x \right]_0^1$$

$$= \frac{1}{4}\left(\frac{1}{3} - 1 + 1 \right) = \boxed{\frac{1}{12}}$$

4. $\boxed{\text{5 points}}$ Compute the following integral using **polar co-ordinates.**

5

$$\int\int_D f(x,y),\ dA, \quad \text{where} \quad f(x,y) = (x^2+y^2)^{-\frac{3}{2}} \text{ and } D: x^2+y^2 \le 1, x+y \ge 1.$$

$y \ge 1 - x$

the line $x + y = 1$

$r\cos\theta + r\sin\theta = 1$

$r = \dfrac{1}{\cos\theta + \sin\theta} = (\cos\theta + \sin\theta)^{-1}$

$(r^2)^{3/2} = r^3$

$$\int_0^{\frac{\pi}{2}} \int_{(\cos\theta+\sin\theta)^{-1}}^{1} (r^2)^{-\frac{3}{2}}\, r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \int_{(\cos\theta+\sin\theta)^{-1}}^{1} r^{-2}\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \Big[-r^{-1}\Big]_{(\cos\theta+\sin\theta)^{-1}}^{1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} -1 - \big(-(\cos\theta + \sin\theta)\big)\,d\theta = \int_0^{\frac{\pi}{2}} (\cos\theta + \sin\theta - 1)\,d\theta$$

$$= \Big[\sin\theta - \cos\theta - \theta\Big]_0^{\frac{\pi}{2}} = 1 - 0 - \frac{\pi}{2} - (0 - 1 - 0)$$

$$= 2 - \frac{\pi}{2}$$