Midterm 2

Last Name: Will's Solutions

First Name:

Student ID:

Signature:

Section:

	Tuesday:	Thursday:	
	3A	3B	TA: Ioannis Lagkas-Nikolos
	3C	3D	TA: Fei Xie
	3E	3F	TA: Sangchul Lee

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Problem	Max	Score
1	10	
2	10	
3	10	
4	10	
Total	40	

1. (10 pts) Suppose a wire is wound along a cone in such a way that its path is described by

$$\vec{r}(t) = \langle t^2, t\cos t, t\sin t\rangle, \qquad 1 \le t \le 4$$

If the charge density (per unit length) along the wire is given by

$$\rho(x,y,z) = \frac{y^2+z^2}{\sqrt{x}},$$

find the total charge in the wire.

Want to compute $\int_C \rho(x,y,z)\,ds$. Recall that $ds = \|\vec{r}\,'(t)\|dt$.

$\vec{r}\,'(t) = \langle 2t,\ \cos t - t\sin t,\ \sin t + t\cos t \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{4t^2 + (\cos^2 t - \cancel{2t\sin t\cos t} + t^2\sin^2 t) + (\sin^2 t + \cancel{2t\sin t\cos t} + t^2\cos^2 t)}$

$= \sqrt{4t^2 + (\cos^2 t + \sin^2 t) + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5t^2+1}$

$\rho(t^2, t\cos t, t\sin t) = \dfrac{(t\cos t)^2 + (t\sin t)^2}{\sqrt{t^2}} = \dfrac{t^2}{t} = t$

So total charge $= \int_C \rho(x,y,z)\,ds = \int_{t=1}^{4} t\cdot\sqrt{5t^2+1}\,dt$

$u = 5t^2+1$
$du = 10t\,dt$
$dt = \dfrac{du}{10t}$

$t=1 \Rightarrow u=6$
$t=4 \Rightarrow u=81$

$= \int_{u=6}^{81} \cancel{t}\sqrt{u}\cdot\dfrac{du}{10\cancel{t}} = \dfrac{1}{10}\left[\dfrac{2}{3}u^{3/2}\right]_{u=6}^{81}$

$= \boxed{\dfrac{1}{15}\left(81^{3/2} - 6^{3/2}\right)}$

$= \dfrac{1}{15}(81\cdot 9 - 6\sqrt{6}) = \boxed{\dfrac{243-2\sqrt{6}}{5}}$

Extraneous information!

2. (10 pts) Let C be a helix of radius 3, centered along the z-axis, that makes 17 complete rotations clockwise while going up 4 units in the z direction, starting from the point $(0,3,1)$. Compute the following line integral:

$$\int_C \sin y\,dx + \left(x\cos y - \frac{1}{z}\right)dy + \frac{y}{z^2}\,dz.$$

(Hint: You don't need to parametrize C.)

Vector field $\vec{F}(x,y,z) = \langle \sin y,\ x\cos y - \frac{1}{z},\ \frac{y}{z^2} \rangle$

$$\left.\begin{aligned} \int \sin y\,dx &= x\sin y + C_1(y,z) \\ \int \left(x\cos y - \tfrac{1}{z}\right)dy &= x\sin y - \tfrac{y}{z} + C_2(x,z) \\ \int \tfrac{y}{z^2}\,dz &= -\tfrac{y}{z} + C_3(x,y) \end{aligned}\right\} \boxed{f(x,y,z) = x\sin y - \frac{y}{z}}$$

is a potential function for $\vec{F}$

(Note: $z > 0$ on the whole curve, so this is okay.)

Starting point: $(0,3,1)$
Ending point: $(0,3,5)$

So, by the Fundamental Theorem of Conservative Vector Fields,

$$\int_C \vec{F}\cdot d\vec{s} = \int \sin y\,dx + \left(x\cos y - \tfrac{1}{z}\right)dy + \tfrac{y}{z^2}\,dz = f(0,3,5) - f(0,3,1)$$

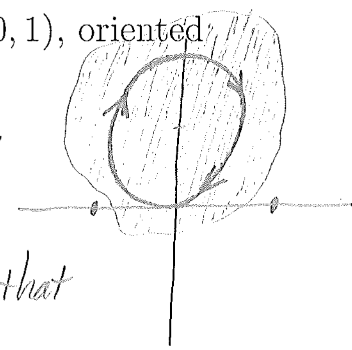
$$= \left(0\cdot\sin(3) - \frac{3}{5}\right) - \left(0\cdot\sin(3) - \frac{3}{1}\right)$$

$$= 3 - \frac{3}{5} = \boxed{\frac{12}{5}}$$

3. (10 pts) Suppose $\vec{F}(x,y) = \langle F_1(x,y), F_2(x,y)\rangle$ is a vector field that is defined and differentiable everywhere in the xy-plane except at the points $(1,0)$ and $(-1,0)$. Suppose you know that, at every point in its domain,

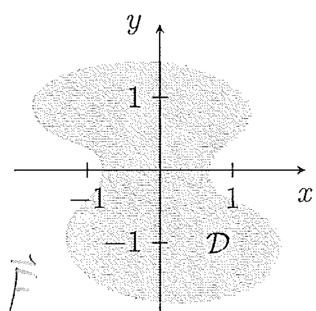
$$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

(a) Let C be the circle of radius 1 centered at the point $(0,1)$, oriented clockwise. Compute $\oint_C \vec{F}\cdot d\vec{s}$. *Justify your answer!*



Since $\vec{F}$ satisfies the "cross partials condition" $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ everywhere on the simply connected domain shown at right, it is conservative on that domain. Therefore $\oint_C \vec{F}\cdot d\vec{s} = \boxed{0}$.

(b) Is $\vec{F}$ conservative on the domain $\mathcal{D}$ shown to the right? *Explain briefly.*



YES

NO

NOT ENOUGH INFO

The domain D is simply connected, and $\vec{F}$ satisfies the cross partials condition everywhere on this domain, so $\vec{F}$ is conservative on this domain.

(c) Is $\vec{F}$ conservative on its whole domain? *Explain briefly.*

YES

NO

NOT ENOUGH INFO

Because $\vec{F}$ is undefined at $(1,0)$ and $(-1,0)$, but defined everywhere around these points, its domain is not simply connected. Therefore we cannot use the theorem we used above to conclude that it is conservative. However, this also does not automatically mean that it isn't conservative. So we do not have enough information to tell for sure.

Hyperboloid (of one sheet)... not necessary to know this, though.

4. (10 pts) Let $\mathcal{S}$ be the portion of the surface $x^2 + y^2 - z^2 = 1$ where $0 \le z \le 3$, oriented with upward-pointing normal vectors. Compute the flux of the vector field

$$\vec{F}(x, y, z) = \langle xz, yz, z^2 \rangle$$

across the surface $\mathcal{S}$.

$x^2+y^2-z^2=1 \iff z^2 = x^2+y^2-1$, and since $z \ge 0$, this means $z = \sqrt{x^2+y^2-1}$. ← Surface

Also $0 \le z \le 3$ means $0 \le z^2 \le 9$, so $0 \le x^2+y^2-1 \le 9$, so $1 \le x^2+y^2 \le 10$ ← Bounds on x,y

One way: Using polar coords

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = \sqrt{r^2-1} \end{cases}$$

$1 \le r^2 \le 10$, so

$1 \le r \le \sqrt{10}$

$0 \le \theta \le 2\pi$

$$\vec{T}_r = \langle \cos\theta, \sin\theta, \tfrac{1}{2}(r^2-1)^{-1/2}\cdot 2r \rangle = \langle \cos\theta, \sin\theta, \tfrac{r}{\sqrt{r^2-1}} \rangle$$

$$\vec{T}_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$\vec{n} = \vec{T}_r \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & \frac{r}{\sqrt{r^2-1}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left\langle \frac{-r^2\cos\theta}{\sqrt{r^2-1}}, \frac{-r^2\sin\theta}{\sqrt{r^2-1}}, r \right\rangle$$

$$= r\left\langle \frac{-r\cos\theta}{\sqrt{r^2-1}}, \frac{-r\sin\theta}{\sqrt{r^2-1}}, 1 \right\rangle$$

Positive z part, so upward! ✓

$$= r\left\langle -\frac{x}{z}, -\frac{y}{z}, 1 \right\rangle$$

Not necessary, but a nice shortcut!

$$\vec{F}\cdot\vec{n} = \langle xz, yz, z^2 \rangle \cdot r\left\langle \frac{-x}{z}, \frac{-y}{z}, 1 \right\rangle$$

$$= r(-x^2-y^2+z^2) = r(-1) = -r$$

because on this surface, $-x^2-y^2+z^2 = -1$

$$\text{So flux} = \iint_S \vec{F}\cdot d\vec{S} = \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{10}} \vec{F}\cdot\vec{n}\,dr\,d\theta = \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{10}} -r\,dr\,d\theta$$

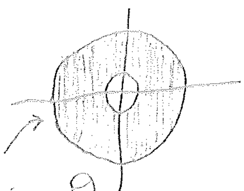
$$= -\int_{\theta=0}^{2\pi}\left(\tfrac{1}{2}r^2\Big|_{r=1}^{\sqrt{10}}\right)d\theta = -\int_{\theta=0}^{2\pi}\left(\tfrac{10}{2}-\tfrac{1}{2}\right)d\theta = -\tfrac{9}{2}\cdot 2\pi = \boxed{-9\pi}$$

Another way: Without using polar coordinates

$$\begin{cases} x = x \\ y = y \\ z = \sqrt{x^2+y^2-1} \end{cases}$$

$$1 \le x^2+y^2 \le 10$$

Note: Domain looks like



Call this D

$$\vec{T}_x = \left\langle 1, 0, \tfrac{1}{2}(x^2+y^2-1)^{-1/2}\cdot 2x \right\rangle = \left\langle 1, 0, \frac{x}{\sqrt{x^2+y^2-1}} \right\rangle$$

$$\vec{T}_y = \left\langle 0, 1, \tfrac{1}{2}(x^2+y^2-1)^{-1/2}\cdot 2y \right\rangle = \left\langle 0, 1, \frac{y}{\sqrt{x^2+y^2-1}} \right\rangle$$

$$\vec{n} = \vec{T}_x \times \vec{T}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{x}{\sqrt{x^2+y^2-1}} \\ 0 & 1 & \frac{y}{\sqrt{x^2+y^2-1}} \end{vmatrix}$$

$$= \left\langle \frac{-x}{\sqrt{x^2+y^2-1}}, \frac{-y}{\sqrt{x^2+y^2-1}}, 1 \right\rangle$$

Positive z part, so upward. ✓

$$\text{So } \vec{F}\cdot\vec{n} = \langle xz, yz, z^2 \rangle \cdot \left\langle \frac{-x}{\sqrt{x^2+y^2-1}}, \frac{-y}{\sqrt{x^2+y^2-1}}, 1 \right\rangle$$

$$= \left\langle x\cancel{\sqrt{x^2+y^2-1}}, y\cancel{\sqrt{x^2+y^2-1}}, x^2+y^2-1 \right\rangle \cdot \left\langle \frac{-x}{\cancel{\sqrt{x^2+y^2-1}}}, \frac{-y}{\cancel{\sqrt{x^2+y^2-1}}}, 1 \right\rangle$$

$$= -\cancel{x^2} - \cancel{y^2} + (\cancel{x^2} + \cancel{y^2} - 1) = -1$$

$$\text{So flux} = \iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\vec{n}\,dx\,dy = \iint_D (-1)\,dx\,dy = -\iint_D 1\,dx\,dy$$

$$= -(\text{area of } D) = -\left(\pi\cdot\sqrt{10}^2 - \pi\cdot 1^2\right) = -(10\pi - \pi)$$

$$= \boxed{-9\pi}$$