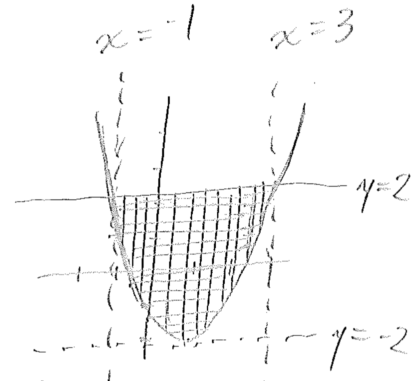
1. (10 pts) Compute the double integral

$$\int_{-2}^{2}\int_{1-\sqrt{2+y}}^{1+\sqrt{2+y}} x\,dx\,dy.$$



(Hint: First change the order of integration.)

$x = 1+\sqrt{2+y} \Rightarrow (x-1)^2 = 2+y$

$x = 1-\sqrt{2+y} \Rightarrow (x-1)^2 = 2+y \Longrightarrow y = (x-1)^2 - 2 = x^2 - 2x - 1$

Parabola w/ vertex at $(1,-2)$

$y = -2 \Rightarrow x = 1$ on parabola (vertex)

$y = 2 \Rightarrow 2 = (x-1)^2 - 2 \Longrightarrow (x-1)^2 = 4,\ x-1 = \pm 2 \Rightarrow x = -1, 3$

So x goes from -1 to 3, y goes from parabola x^2-2x-1 to 2.

$$\int_{-1}^{3}\int_{x^2-2x-1}^{2} x\,dy\,dx = \int_{-1}^{3}\left(xy\Big|_{y=x^2-2x-1}^{y=2}\right)dx$$

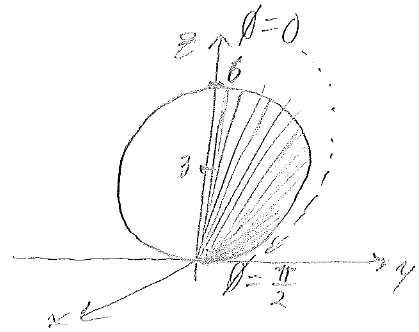
$$= \int_{-1}^{3}\left(2x - x(x^2-2x-1)\right)dx$$

$$= \int_{-1}^{3}\left(-x^3+2x^2+3x\right)dx = -\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{3}{2}x^2\Big|_{-1}^{3}$$

$$= \left(\frac{-81}{4} + \frac{2\cdot 27}{3} + \frac{3\cdot 9}{2}\right) - \left(\frac{-1}{4} - \frac{2}{3} + \frac{3}{2}\right) = \frac{-80}{4} + 18 + \frac{2}{3} + \frac{24}{2}$$

$$= 10 + \frac{2}{3} = \boxed{\frac{32}{3}}$$

$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$



2. (10 pts) Find the average distance from the origin of all points inside the sphere

$$x^2 + y^2 + (z-3)^2 = 9.$$

$\leftarrow$ Sphere of radius 3, centered at (0,0,3)

$x^2+y^2+z^2-6z+\cancel{9}=\cancel{9}$

$\underbrace{x^2+y^2+z^2}-6z=0$ Spherical coords!

$\rho^2 - 6\rho\cos\phi = 0$

$\rho(\rho - 6\cos\phi) = 0 \implies \cancel{\rho=0}$ $(\rho = 6\cos\phi)$

Just a point

Use bounds $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{2}$, $0 \le \rho \le 6\cos\phi$

Distance of a point from origin $= \rho$ $\leftarrow$ Find average value of this

Average value $= \dfrac{\iiint_W \rho \, dV}{\text{Volume of sphere}}$

Note that volume of the sphere is $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot 3^3 = (36\pi)$

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{6\cos\phi} \rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}} \left(\frac{1}{4}\rho^4 \sin\phi \Big|_{\rho=0}^{6\cos\phi} \right) d\phi \, d\theta = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}} \frac{1}{4} 6^4 \cdot \cos^4\phi \sin\phi \, d\phi \, d\theta$$

$u = \cos\phi$, $du = -\sin\phi \, d\phi$

$$= -\frac{6^4}{4}\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} (\cos\phi)^4(-\sin\phi \, d\phi) \, d\theta = -\frac{6^4}{4}\int_{\theta=0}^{2\pi} \frac{1}{5}\cos^5\phi \Big|_{\phi=0}^{\pi/2} d\theta$$

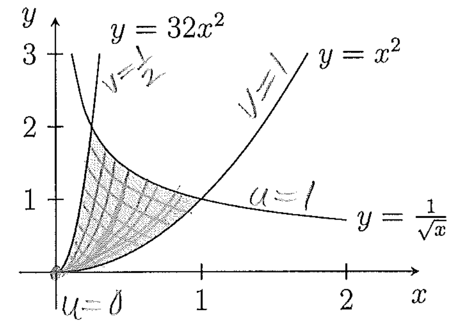
$$= -\frac{\overset{9}{\cancel{36}} \cdot 36}{\cancel{4} \cdot 5}\int_{\theta=0}^{2\pi} (0-1) \, d\theta = \frac{9 \cdot 36}{5}\int_0^{2\pi} 1 \, d\theta = \frac{9 \cdot 36}{5} \cdot 2\pi = \left(\frac{18 \cdot 36\pi}{5}\right)$$

$$\text{Average value} = \frac{\frac{18 \cdot \cancel{36\pi}}{5}}{\cancel{36\pi}} = \boxed{\frac{18}{5}}$$

3. (10 pts) Let $\mathcal{D}$ be the region in the first quadrant bounded by $y = x^2$, $y = 32x^2$, and $y = \frac{1}{\sqrt{x}}$. Use the change of variables $x = uv^2$, $y = \frac{u^2}{v}$ to compute the integral

$$\iint_{\mathcal{D}} ye^{xy^2}\, dA.$$

(Hint: What part of the picture corresponds to $u = 0$?)



$y = x^2$: $\frac{u^2}{v} = (uv^2)^2 \Longrightarrow \frac{\cancel{u^2}}{v} = \cancel{u^2} v^4$

$v^5 = 1$

$v = 1$

$y = 32x^2$: $\frac{\cancel{u^2}}{v} = 32\cancel{u^2} v^4 \Longrightarrow v^5 = \frac{1}{32}$

$v = \frac{1}{2}$

$y = \frac{1}{\sqrt{x}}$: $\frac{u^2}{v} = \frac{1}{\sqrt{uv^2}} \longrightarrow \frac{u^4}{\cancel{v^2}} = \frac{1}{u\cancel{v^2}} \Longrightarrow u^5 = 1$

$u = 1$

$u = 0 \Longrightarrow x = 0,\ y = 0\ldots$ origin!

Jacobian $= \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} v^2 & \frac{2u}{v} \\ 2uv & -\frac{u^2}{v^2} \end{pmatrix} = (\cancel{v^2})\left(-\frac{u^2}{\cancel{v^2}}\right) - (2u\cancel{v})\left(\frac{2u}{\cancel{v}}\right) = -5u^2$

So $\iint_D ye^{xy^2}\,dA = \int_{v=\frac{1}{2}}^{1}\int_{u=0}^{1} \frac{u^2}{v} e^{\left(uv^2 \cdot \frac{u^4}{v^2}\right)} \left|-5u^2\right| du\,dv$

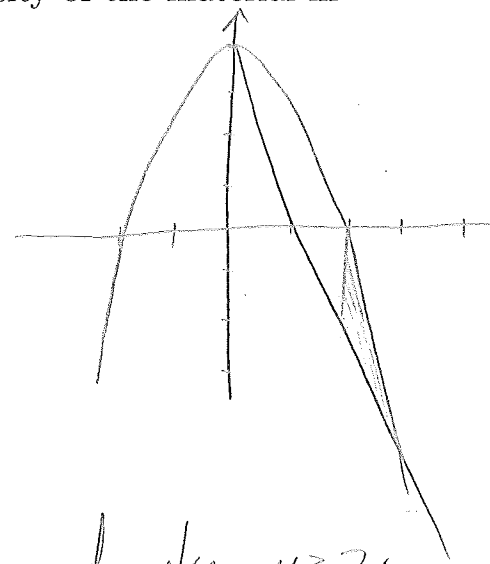
$= \int_{v=\frac{1}{2}}^{1}\int_{u=0}^{1} \frac{5u^4}{v} e^{(u^5)}\,du\,dv = \int_{v=\frac{1}{2}}^{1}\left(\frac{1}{v} e^{(u^5)}\Big|_{u=0}^{1}\right) dv$

$= \int_{v=\frac{1}{2}}^{1} \frac{1}{v}\left(e^1 - e^0\right) dv = (e-1)\cdot \ln(v)\Big|_{v=\frac{1}{2}}^{1}$

$= (e-1)\cdot\left(\overset{0}{\cancel{\ln 1}} - \ln\frac{1}{2}\right) = (e-1)\left(-\ln\frac{1}{2}\right) = \boxed{(e-1)\cdot\ln 2}$

4. (10 pts, 7 for just setup) Find the total mass within the region described by $z \leq 4 - x^2 - y^2$, $z \geq 4 - 4y$, $y \geq 2$, if the density of the material in that region is given by

$$\rho(x, y, z) = \frac{1}{x^2 + y^2}.$$



Projection onto xy-plane:

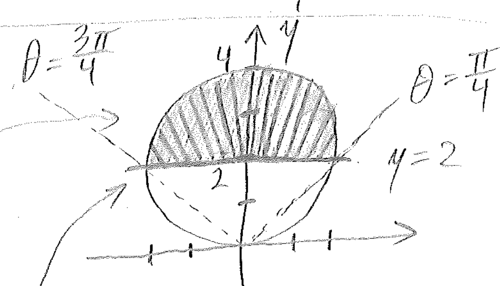
$z = 4 - x^2 - y^2$, $z = 4 - 4y$

$\cancel{4} - x^2 - y^2 = \cancel{4} - 4y$

$x^2 + y^2 - 4y + 4 = 0 + 4$

$x^2 + (y-2)^2 = 4$

Circle centered at $(0,2)$, w/ radius 2 ... and also $y \geq 2$:



In polar: $x^2 + y^2 - 4y = 0$

$r^2 - 4r\sin\theta = 0$

$r(r - 4\sin\theta) = 0$

$\cancel{r=0}$ or $r = 4\sin\theta$

And $y = 2 \Leftrightarrow r\sin\theta = 2 \longrightarrow r = \frac{2}{\sin\theta} = 2\csc\theta$

So $\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$ and $2\csc\theta \leq r \leq 4\sin\theta$, and $4 - 4y \leq z \leq 4 - x^2 - y^2$

So total mass $= \iiint_W \rho(x,y,z)\,dV = \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{r=2\csc\theta}^{4\sin\theta} \int_{z=4-4r\sin\theta}^{4-r^2} \frac{1}{r^2} \cdot r \, dz\,dr\,d\theta$

$= \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{r=2\csc\theta}^{4\sin\theta} \frac{1}{r} \cdot z \Big|_{z=4-4r\sin\theta}^{4-r^2} dr\,d\theta$

$= \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{r=2\csc\theta}^{4\sin\theta} \frac{1}{r}\left((\cancel{4} - r^2) - (\cancel{4} - 4r\sin\theta)\right) dr\,d\theta$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_{r=2\csc\theta}^{4\sin\theta} (4\sin\theta - r)\, dr\, d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} \left[4r\sin\theta - \frac{1}{2}r^2\right]_{r=2\csc\theta}^{4\sin\theta} d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} \left((16\sin^2\theta - 8\sin^2\theta) - (8\csc\theta\sin\theta - 2\csc^2\theta)\right) d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} (8\sin^2\theta + 2\csc^2\theta - 8)\, d\theta$$

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} (4 - 4\cos 2\theta + 2\csc^2\theta - 8)\, d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}} (2\csc^2\theta - 4\cos 2\theta - 4)\, d\theta$$

$$= -2\cot\theta - 2\sin 2\theta - 4\theta \Big|_{\theta=\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$\cot\left(\frac{3\pi}{4}\right) = -1 \quad \sin\left(\frac{3\pi}{2}\right) = -1$

$\cot\left(\frac{\pi}{4}\right) = 1 \quad \sin\left(\frac{\pi}{2}\right) = 1$

$$= (-2\cdot(-1) - 2(-1) - 3\pi) - (-2\cdot 1 - 2\cdot 1 - \pi)$$

$$= 4 - 3\pi + 4 + \pi = \boxed{8 - 2\pi}$$