Math 32B Final Exam

[redacted]

TOTAL POINTS

90 / 100

QUESTION 1

1 Change order of integration 4 / 4

✓ **- 0 pts** **answer = 2 (limits are x=0,pi y=0,x)**

- **- 1 pts** minor error
- **- 2 pts** incorrect integration bounds
- **- 1 pts** integration error
- **- 4 pts** swapping the order of integration without changing the bounds
- **- 4 pts** incorrect
- **- 2 pts** major integration error

QUESTION 2

2 Spherical coords 4 / 8

- 0 pts (1 pt) theta 0 to 2pi
(2 pts) phi pi/6 to 5pi/6
(2 pts) rho lower bound 1/sin phi
(1 pt) rho upper bound 2
(2 pts) integrand rho^2 sin phi

- **- 1 pts** 1 error
- **- 2 pts** 2 errors
- **- 3 pts** 3 errors

✓ **- 4 pts** **4 errors**

- **- 5 pts** 5 errors
- **- 6 pts** 6 errors
- **- 7 pts** 7 errors
- **- 8 pts** 8 errors

QUESTION 3

Vortex field 12 pts

3.1 line integral 4 / 4

✓ **- 0 pts** **2pi**

- **- 2 pts** Incorrect integral setup
- **- 2 pts** Integration error
- **- 1 pts** Minor error
- **- 4 pts** Incorrect

3.2 curlz(F) 3 / 3

✓ **- 0 pts** **0**

- **- 1 pts** minor error
- **- 2 pts** major error
- **- 3 pts** completely incorrect
- **- 1 pts** should be a scalar, not a vector

3.3 Fill in the blanks 2 / 2

✓ **- 0 pts** **0, simply connected**

- **- 1 pts** one wrong
- **- 2 pts** both wrong

3.4 Conservative? 3 / 3

✓ **- 0 pts** **No, because the integral in (a) is nonzero**

- **- 1 pts** No (but partially correct reason)
- **- 2 pts** No (but incorrect reason)
- **- 3 pts** Yes

QUESTION 4

Surface integral w/ vector potential 12 pts

4.1 vector potential 2 / 2

✓ **- 0 pts** **Correct.**

- **- 1 pts** Incorrect, but knew that they needed calculate the curl of A.
- **- 2 pts** Incorrect.

4.2 Stokes' theorem 8 / 8

✓ **+ 2 pts** **Applying Stoke's Theorem**

✓ **+ 2 pts** **Correctly parameterising the boundary.**

✓ **+ 1 pts** **Correct Orientation on boundary**

✓ **+ 2 pts** **Correctly setting the boundary integral up.**

✓ **+ 1 pts** **Correct final answer. (-24\pi or 24\pi if orientation wrong.)**

- **+ 0 pts** Incorrect.

4.3 Other orientation 2 / 2

✓ **- 0 pts** **Correct. (negative of answer in b)**

- 1 pts Almost Correct (same as answer in b)

- 2 pts Incorrect

QUESTION 5

5 Worksheet problem: line integral in a plane 12 / 12

✓ **+ 12 pts** **Correct**

+ 2 pts Stokes' Theorem

+ 2 pts Correct curl

+ 2 pts Correct normal vector / orientation

+ 1 pts normalized

+ 2 pts dot with curl

+ 2 pts Recognizing the surface area as the integral of 1

+ 1 pts Correct answer (15*sqrt(3) or 45/sqrt(3))

+ 0 pts Click here to replace this description.

QUESTION 6

Divergence theorem 12 pts

6.1 integral of bottom cap 4 / 4

✓ **- 0 pts** **Correct**

- 1 pts Incorrect integrand (r^2 instead of r^3)

- 1 pts Incorrect integrand (r^4 instead of r^3)

- 1 pts Sign error

- 1 pts Integration error

- 1 pts Incorrect integrand (r instead of r^3)

- 1 pts Incorrect integrand (should have r^3)

6.2 integral of hemisphere 8 / 8

✓ **- 0 pts** **Correct**

- 0 pts Correct, given your answer to (a)

- 8 pts Incorrect

- 2 pts Incorrect divergence

- 0.5 pts Minor calculation error

- 2 pts Forgot to solve for flux at end

- 1 pts Incorrect integrand

- 1 pts Sign error

- 1 pts Small calculation error

- 2 pts Incorrect integral

- 7 pts 1 point for attempting to write out the integral in terms of a parametrization

QUESTION 7

MC 15 pts

7.1 (a) 0 / 3

✓ **- 3 pts** **Incorrect**

- 0 pts Correct (positive)

7.2 (b) 3 / 3

- 3 pts Incorrect

✓ **- 0 pts** **Correct (positive)**

7.3 (c) 3 / 3

✓ **- 0 pts** **Correct (zero)**

- 3 pts Incorrect

7.4 (d) 0 / 3

✓ **- 3 pts** **Incorrect**

- 0 pts Correct (0.2)

7.5 (e) 3 / 3

✓ **- 0 pts** **Correct (div(curl F)=0)**

- 3 pts Incorrect

- 1.5 pts Click here to replace this description.

- 2 pts Click here to replace this description.

QUESTION 8

MC 15 pts

8.1 (a) 3 / 3

✓ **- 0 pts** **Correct**

- 3 pts Incorrect

8.2 (b) 3 / 3

✓ **- 0 pts** **Correct**

- 3 pts Incorrect

8.3 (c) 3 / 3

✓ **- 0 pts** **Correct**

- **3 pts** Incorrect

similar)

8.4 (d) **3 / 3**

✓ **- 0 pts Correct**

- 3 pts Incorrect

8.5 (e) **3 / 3**

✓ **- 0 pts Correct**

- 3 pts Incorrect

QUESTION 9

Fill in the blanks 10 pts

9.1 counterclockwise **1 / 1**

✓ **- 0 pts Correct**

- 1 pts Incorrect.

9.2 curlz(F) **1 / 1**

✓ **- 0 pts Correct**

- 1 pts Incorrect.

- 0.5 pts Wrote curl(F) instead of curl_z(F), or got the order of derivatives wrong way.

9.3 boundary of D **1 / 1**

✓ **- 0 pts Correct**

- 1 pts Incorrect

9.4 RHS of Stokes' thm **3 / 3**

✓ **- 0 pts Correct**

- 1 pts Incorrect integral bounds (\partial S)

- 1 pts Did not put single integral.

- 1 pts Incorrect integrand

9.5 outward **1 / 1**

✓ **- 0 pts Correct**

- 1 pts Incorrect

9.6 LHS of Div thm **3 / 3**

✓ **- 0 pts Correct**

- 1 pts Not triple integral

- 1 pts Bounds wrong (W)

- 1 pts Integrand wrong. (div(F)dV, div(F)dxdydz or

MATH 32B Final Exam Spring 2019

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Section	3	A

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- If you have a question, raise your hand and one of the proctors will come to you. We will not answer any mathematical questions except possibly to clarify the wording of a problem.
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Spherical coordinates:

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$
$$z = \rho \cos\phi$$

$$dxdydz = \rho^2 \sin\phi \, d\rho d\phi d\theta$$

Page:	1	2	3	4	5	6	7	8	Total
Points:	12	12	12	12	12	15	15	10	100
Score:									

1. (4 points) Evaluate the integral $\int_0^\pi \int_y^\pi \frac{\sin x}{x}\, dx\, dy$ by changing the order of integration.

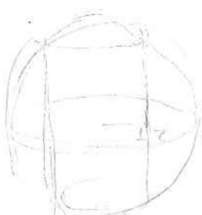
$y \le x \le \pi$
$0 \le y \le \pi$ $\rightarrow$ $0 \le y \le x$
$0 \le x \le \pi$

$$\int_0^\pi \int_0^x \frac{\sin x}{x}\, dy\, dx$$

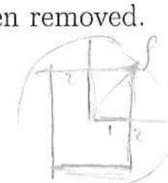
$$= \int_0^\pi \left(\frac{\sin x}{x} y \Big|_0^x \right) dx = \int_0^\pi \left(\frac{\sin x}{\cancel{x}}(\cancel{x}) - \frac{\sin x}{\cancel{x}}(0)^{\to 0} \right) dx$$

$$= \int_0^\pi \sin x\, dx = -\cos x \Big|_0^\pi = -\cos(\pi) + \cos(0) = 1 + 1 = \boxed{2}$$

2. (8 points) Using spherical coordinates, set up *but do not evaluate* a triple integral that computes the volume of a sphere of radius 2 from which a central cylinder of radius 1 has been removed.



$0 \le \theta \le 2\pi$
$1 \le \rho \le 2$
$0 \le \phi \le \pi$



$$\iiint_W dV = \boxed{\int_0^{2\pi} \int_0^{\pi} \int_1^{2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$$

3. (12 points) Let $\boldsymbol{F}$ denote the vortex field $\boldsymbol{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

(a) Suppose that C_R is the circle of radius R centered at $(0,0)$ oriented counterclockwise.
By parametrizing C_R, compute $\oint_{C_R} \boldsymbol{F} \cdot d\boldsymbol{r}$. $\boxed{\text{Box your answer}}$
Note: you may not use the fundamental theorem of line integrals or anything about the winding number in this problem. Also, the answer is not zero.

$C_R : r(t) = \langle R\cos t, R\sin t \rangle \qquad r'(t) = \langle -R\sin t, R\cos t \rangle$

$x^2+y^2 = R^2$

$$\int_{C_R} \vec{F}\cdot d\vec{r} = \int_{t_0}^{t} \vec{F}(\vec{r}(t)) \cdot r'(t)\,dt = \int_0^{2\pi} \left\langle \frac{-R\sin t}{R^2}, \frac{R\cos t}{R^2} \right\rangle \cdot \langle -R\sin t, R\cos t \rangle\,dt$$

$$= \int_0^{2\pi} \frac{\cancel{R^2}\sin^2 t}{\cancel{R^2}} + \frac{\cancel{R^2}\cos^2 t}{\cancel{R^2}}\,dt = \int_0^{2\pi} \sin^2 t + \cos^2 t\,dt$$

$$= \int_0^{2\pi} 1\,dt = t\Big|_0^{2\pi} = \boxed{2\pi}$$

(b) Compute $\text{curl}_z(\boldsymbol{F})$. Show your work. $\boxed{\text{Box your answer}}$

$$\text{curl}_z(\vec{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \left(\frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2}\right) - \left(\frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}\right)$$

$$= \frac{\cancel{x^2} + \cancel{y^2} - \cancel{2x^2} + \cancel{x^2} + \cancel{y^2} - \cancel{2y^2}}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2}$$

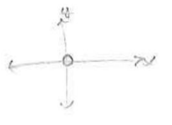
$$= \boxed{0}$$

(c) Fill in the blanks:

(i) If $\boldsymbol{F} = \nabla f$ on a domain D then $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} =$ ___0___ for every closed curve C in D.

(ii) If $\text{curl}_z(\boldsymbol{F}) = 0$ on a ___simply connected___ domain D then $\boldsymbol{F}$ is conservative.

(d) Is the vortex field $\boldsymbol{F}$ conservative on the domain $\mathbb{R}^2 \setminus \{(0,0)\}$? Explain your reasoning.



- No, the vortex field is not conservative on this domain.
- The vortex field's domain is not "simply connected," thus even if the curl_z is 0, we cannot say it is conservative.
- Vortex field does not have a potential function f, such that $\nabla f = \vec{F}$. Even when creating a "potential" for it, one of the axes (usually $-x$ axis) must slice through the domain

$\cancel{\nabla f} = \vec{F}$

- Vortex field is not path independent. Closed paths around the origin yield 2π for each loop, while closed paths not around $(0,0)$ evaluate to 0.

4. (12 points) Let $\boldsymbol{F} = \langle 2x, 0, -2z\rangle$.

(a) Verify that $\boldsymbol{A} = \langle yz, -xz, yx\rangle$ is a vector potential for $\boldsymbol{F}$.

$\vec{F} = \text{curl}(\vec{A})$?

$$\text{curl}\,\vec{A} = \nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xz & yx \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yz & -xz \end{matrix}$$

$$= \langle x-(-x),\ y-y,\ -z-z\rangle$$

$$= \boxed{\langle 2x, 0, -2z\rangle}$$

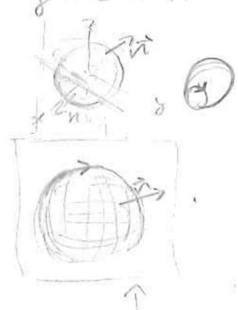
(b) Let $\mathcal{S}$ be the portion of the sphere $x^2 + y^2 + z^2 = 13$ where $x \le 3$, oriented with outward-pointing normal vector. Find the flux of $\boldsymbol{F}$ through $\mathcal{S}$. *Hint:* use the result of part (a). $\boxed{\text{Box your answer}}$

Stokes' Thm.

$x^2+y^2+z^2=13$

@$x=3 \to 9+y^2+z^2=13$

$y^2+z^2=4$



$$\text{flux} = \iint_S \vec{F}\, d\vec{S} = \iint_S \text{curl}\,\vec{A}\, d\vec{S} = \int_{\partial S} \vec{A}\cdot d\vec{r}$$

$$= \int_{t_0}^{t} \vec{A}(\vec{r}(t))\cdot\vec{r}\,'(t)\, dt$$

$\vec{A}(r(t)) = \langle 4\cos t\sin t, -6\sin t, 6\cos t\rangle$

$$= -\int_0^{2\pi} \langle 4\cos t\sin t, -6\sin t, 6\cos t\rangle \cdot \langle 0, -2\sin t, 2\cos t\rangle\, dt$$

$$= -\int_0^{2\pi} (0 + 12\sin^2 t + 12\cos^2 t)\, dt$$

$$= -\int_0^{2\pi} 12\, dt = 12t\Big|_0^{2\pi} = -(12(2\pi) - 12(0))$$

$$= \boxed{-24\pi}$$

$x=3$

Boundary:

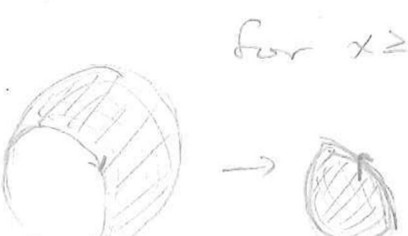
$\vec{r}(t) = \langle 3, 2\cos t, 2\sin t\rangle$

$\vec{r}\,'(t) = \langle 0, -2\sin t, 2\cos t\rangle$

For $x \le 3$, we used this negative because the boundary is oriented clockwise, but we parametrized $r(t)$ counterclockwise.

(c) Let $\mathcal{S}'$ be the portion of the sphere $x^2 + y^2 + z^2 = 13$ where $x \ge 3$, oriented with outward-pointing normal vector. Find the flux of $\boldsymbol{F}$ through $\mathcal{S}'$. $\boxed{\text{Box your answer}}$
You don't need to show your work for this part of the problem.

for $x \ge 3$, the boundary orientation is flipped!



$$\iint_{S'} \vec{F}\, d\vec{S} = -\iint_S \vec{F}\, d\vec{S} = \boxed{24\pi}$$

5. (12 points) Given that C is a simple closed curve in the plane $x+y+z=1$ (oriented counterclockwise when viewed from above) that encloses a surface area of 5, compute $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for $\boldsymbol{F} = \langle 3z, 2x, 4y \rangle$.

$\boxed{\text{Box your answer}}$ *Hint:* it may be helpful to remember that $\iint_S \boldsymbol{G} \cdot d\boldsymbol{S} = \iint_S (\boldsymbol{G} \cdot \boldsymbol{n}) dS$.

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl}(\vec{F}) \cdot d\vec{S}$$

$$\underline{\text{curl}\,\vec{F}} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3z & 2x & 4y \end{vmatrix} \begin{matrix} \hat{i} & \hat{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 3z & 2x \end{matrix}$$

$$= \langle 4-0, 3-0, 2-0 \rangle$$

$$= \underline{\langle 4, 3, 2 \rangle}$$

$\underline{\hat{n}}$ for $x+y+z=1$

$\sqrt{1^2+1^2+1^2} = \sqrt{3}$

$$\hat{n} = \frac{\vec{N}}{\|\vec{N}\|} = \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \underbrace{\langle 4, 3, 2 \rangle}_{\nabla \times \vec{F}} \cdot \underbrace{\left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle}_{\hat{n}} dS$$

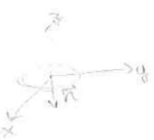
surface area

$$= \frac{9}{\sqrt{3}} \iint_S dS = \frac{{}^{3}\cancel{9}\sqrt{3}}{{}_{1}\cancel{3}} \iint_S dS = (3\sqrt{3})(5) = \boxed{15\sqrt{3}}$$

6. (12 points) Let $\boldsymbol{F} = \langle z^2x, \frac{1}{3}y^3 + \sin^2 z, x^2z + y^2\rangle$.

(a) Let $\mathcal{D}$ be the unit disk $x^2+y^2 \le 1$ in the xy-plane, oriented downward. Compute $\iint_{\mathcal{D}} \boldsymbol{F}\cdot d\boldsymbol{S}$.

It may be helpful to know that $\int_0^{2\pi} \sin^2\theta\, d\theta = \pi$. $\boxed{\text{Box your answer}}$
Hint: if $\mathcal{D}$ is parametrized via $G(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$ then $\boldsymbol{N} = \pm\langle 0,0,r\rangle$.



$$\iint_D \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \vec{F}(G(r,\theta))\cdot \vec{N}(r,\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left\langle 0, \frac{r^3\sin^3\theta}{3}, r^2\sin^2\theta\right\rangle \cdot \langle 0, 0, -r\rangle$$

$$= \int_0^{2\pi}\int_0^1 0 + 0 - r^3\sin^2\theta\, dr\, d\theta = -\int_0^{2\pi}\left(\frac{r^4}{4}\Big|_0^1\right)\sin^2\theta\, d\theta$$

$$= -\int_0^{2\pi}\left(\frac{1}{4} - \cancel{\frac{0}{4}}\right)\sin^2\theta\, d\theta = -\frac{1}{4}\int_0^{2\pi}\sin^2\theta\, d\theta = -\frac{1}{4}(\pi) = \boxed{-\frac{\pi}{4}}$$

(b) Let $\mathcal{S}$ be the top half of the sphere $x^2+y^2+z^2 = 1$, oriented upward. Compute $\iint_{\mathcal{S}} \boldsymbol{F}\cdot d\boldsymbol{S}$.

$\boxed{\text{Box your answer}}$ *Hint:* you should use your answer to part (a). If you cannot do part (a), let A denote the value of the integral in part (a) and give your answer in terms of A.

∂W: S + unit disk

(keeping D oriented downward gives it an outward $\vec{n}$ when part of ∂W)

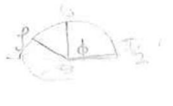
$$\int_{\partial C}\omega = \int_C \partial\omega \to \iint_{\partial W} \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\, dV$$

$$\operatorname{div}(\vec{F}) = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}\left(\frac{y^3}{3} + \sin^2 z\right) + \frac{\partial}{\partial z}(x^2z + y^2)$$

$$= z^2 + y^2 + x^2$$

$\rho^2 = x^2+y^2+z^2$

$$\iiint_W x^2+y^2+z^2\, dV = \iiint_W x^2+y^2+z^2\, dx\, dy\, dz$$



$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho^2)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \left(\frac{\rho^5}{5}\Big|_0^1\right)\left(-\cos\phi\Big|_0^{\pi/2}\right)\left(\theta\Big|_0^{2\pi}\right)$$

$$= \left(\frac{1}{5} - 0\right)\left(-\cancel{\cos\left(\frac{\pi}{2}\right)} + \cos(0)\right)(2\pi - \cancel{0})$$

$\partial W = S$ + unit disk (D)

so,

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{\partial W}\vec{F}\cdot d\vec{S} - \iint_D \vec{F}\cdot d\vec{S}$$

$$= \left(\frac{1}{5}\right)(0+1)(2\pi)$$

$$= \frac{2\pi}{5}$$

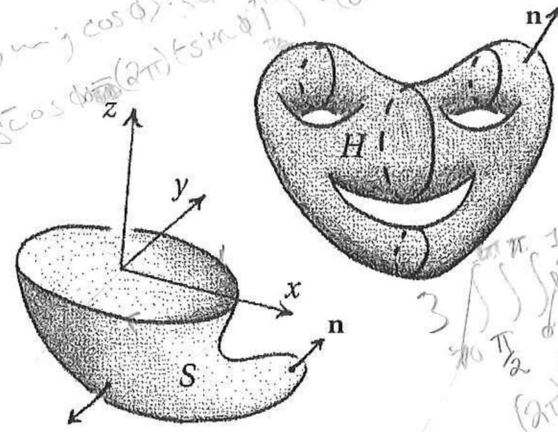
$$\to \iint_S \vec{F}\cdot d\vec{S} = \frac{2\pi}{5} - \left(-\frac{\pi}{4}\right) = \frac{2\pi}{5} + \frac{\pi}{4}$$

$$= \frac{8\pi}{20} + \frac{5\pi}{20} = \boxed{\frac{13\pi}{20}}$$

7. (15 points) Multiple choice. Circle the correct answer.

Let S and H be the surfaces shown to the right. The boundary of S is the unit circle in the xy-plane, while H has no boundary. Let $\boldsymbol{G} = \langle x, y, z \rangle$.

$f = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$



(a) The flux $\iint_H \boldsymbol{G} \cdot d\boldsymbol{S}$ is

$\iint_S \vec{F} \cdot d\vec{S}$ over a closed surface $= 0$

negative zero positive

(b) The flux $\iint_S \boldsymbol{G} \cdot d\boldsymbol{S}$ is

negative zero positive

Hint for (b): use the divergence theorem.

$= \iiint_W \operatorname{div}(G)\, dV - \iint \vec{G} \cdot d\vec{r}$

$= 3 \text{ volume}(W)$

$\operatorname{div}(G) = 1 + 1 + 1 = \boxed{3}$

(c) The flux $\iint_S \operatorname{curl} \boldsymbol{G} \cdot d\boldsymbol{S}$ is

closed loop

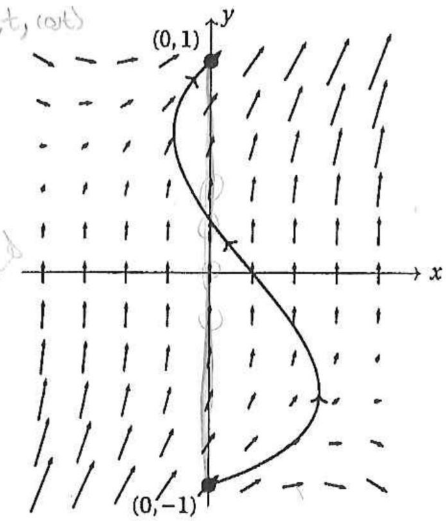
negative zero positive

$= -\oint_{\partial S} \vec{G} \cdot d\vec{r} = -\int_0^{2\pi} \langle \cos t, \sin t, 0 \rangle \cdot \langle -\sin t, \cos t \rangle$

$= -\sin t \cos t + \sin t \cos t$

$= 0$

(d) A vector field is shown to the right. For scale, $\boldsymbol{F}(0,0) = \langle 0, 0.1 \rangle$.



Given that $\underline{\boldsymbol{F} \text{ is conservative}}$, estimate $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C is the curve shown from $(0, -1)$ to $(0, 1)$.

path independent

-0.5 $\quad -0.2$ $\quad 0$ $\quad 0.2$ $\quad 0.5$

positive

(e) Which of the following statements makes sense and is true for any vector field $\boldsymbol{F}$ in $\mathbb{R}^3$ whose components have continuous second-order partial derivatives?

$f \to \nabla f \to \nabla \times \vec{F} \to \nabla \cdot \vec{F} \to g$

$\nabla(\operatorname{curl} \boldsymbol{F}) = 0$ $\quad \operatorname{div}(\operatorname{curl} \boldsymbol{F}) = 0$ $\quad \operatorname{div}(\nabla \boldsymbol{F}) = 0$ $\quad \operatorname{curl}(\operatorname{curl} \boldsymbol{F}) = 0$

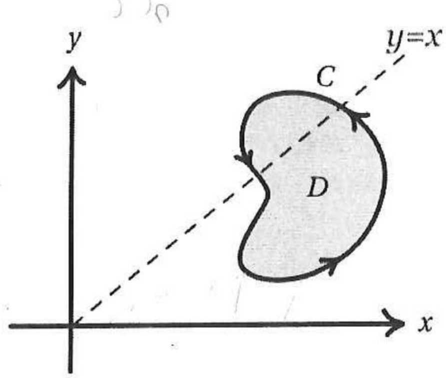
$\operatorname{curl}(\nabla f) = 0$

$\operatorname{div}(\operatorname{curl} \vec{F}) = 0$

8. (15 points) Multiple choice. Circle the correct answer.
Consider the region D in the plane bounded by the curve C as shown to the right. For parts (a)–(c), circle the best answer.

B). $\iint_D 0+1 \, dA = \iint_D dA =$ positive

a). $\iint_D 0-0 \, dA = \iint_D 0 \, dA = 0$



(a) For $\boldsymbol{F}(x, y) = \langle x+1, y^2 \rangle$, the integral $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$ is

negative   zero   positive

(b) The integral $\int_C (-y\,dx + 2dy)$ is

$\vec{F} = \langle -y, 2 \rangle$

negative   zero   positive

(c) The integral $\iint_D (y-x)\, dA$ is

negative   zero   positive

Hint for (c): look at the location of D in the plane.

more negative stuff on x-axis

c). $\int_C \langle xy, xy \rangle \, d\vec{r}$

(d) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation of the plane sending the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ to the triangle with vertices $(0,0)$, $(1,2)$, $(-1,3)$, respectively. Find the Jacobian of T.

1   2   3   4   5

$G(u,v) = (au+bv, cu+dv) \Rightarrow G(1,0) = (a,c) = (1,2)$

$G(0,1) = (b,d) = (-1,3)$

$G(u,v) = (u-v, 2u+3v)$

$\mathrm{Jac}(G) = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$

$3+2 = 5$

(e) Let $\mathcal{R} = [1,2] \times [1,2]$ and let $\mathcal{D} = G(\mathcal{R})$, where G is the map $G(u,v) = (u^2/v, v^2/u)$. Compute the area of $\mathcal{D}$.

1   2   3   4   5

area(R) = 1

$1 \cdot 3 = \boxed{3}$

$\mathrm{Jac}(G) = \begin{vmatrix} \frac{2u}{v} & -\frac{u^2}{v^2} \\ -\frac{v^2}{u^2} & \frac{2v}{u} \end{vmatrix}$

$\frac{4uv}{uv} - \frac{v^2u^2}{u^2v^2}$

$4 - 1 = \underline{3}$

9. (10 points) Fill in the blanks in the big theorems of vector calculus.

The fundamental theorem of line integrals. If $\mathcal{C}$ is an oriented curve from P to Q in $\mathcal{D}$ then

$$\int_{\mathcal{C}} \nabla f \cdot d\boldsymbol{r} = f(Q) - f(P).$$

Green's theorem. Let $\mathcal{D}$ be a domain whose boundary ∂D is a simple closed curve, oriented $\boxed{\text{counterclockwise}}$. Then

$$\iint_{\mathcal{D}} \boxed{\mathrm{curl}_z(\vec{F})}\, dA = \oint_{\boxed{\partial D}} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

$\uparrow$

if $\vec{F} = \langle F_1, F_2 \rangle$, $\mathrm{curl}_z(\vec{F}) = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$

Stokes' theorem. Let $\mathcal{S}$ be a "sufficiently nice" surface, and let $\boldsymbol{F}$ be a vector field whose components have continuous partial derivatives on an open region containing $\mathcal{S}$. Then

$$\iint_{\mathcal{S}} \mathrm{curl}(\boldsymbol{F}) \cdot d\boldsymbol{S} = \boxed{\oint_{\partial S} \vec{F} \cdot d\vec{r}}.$$

The integral on the right-hand side is defined relative to the boundary orientation of ∂S.

The divergence theorem. Let $\mathcal{S}$ be a closed surface that encloses a region $\mathcal{W}$ in $\mathbb{R}^3$. Assume that $\mathcal{S}$ is piecewise smooth and is oriented by normal vectors pointing $\boxed{\text{outward}}$. Let $\boldsymbol{F}$ be a vector field whose domain contains $\mathcal{W}$. Then

$$\boxed{\iiint_W \mathrm{div}(\vec{F})\, dV} = \iint_{\partial \mathcal{W}} \boldsymbol{F} \cdot d\boldsymbol{S}.$$

You may use this page for scratch work.