Math 32A/4
Fall 2012
Exam 2
November 19, 2012
100 points

Name (Print): ____________________

ID #: ____________________

TA: ____________________

This exam consists of 7 pages (including this cover page) and 4 problems. Please check to see if any pages are missing.

Problem	Points	Score
1	25	
2	25	
3	25	
4	25	
Total:	100	

- **Rules.** No calculators, computers, notes, books, or other aids are allowed.

- **Style.** To receive full credit, the reasoning leading to a solution must be clear and complete. A correct answer given without a complete and correct argument will be worth little or no credit.

  Organize your work. Messy and scattered work without a clear order will receive very little credit.

  In the statements of the problems, vectors are set in boldface ($\mathbf{v}$) and scalars are in plain type (v). In your solutions, please write vectors with arrows ($\overrightarrow{v}$) and scalars without arrows (v).

  There is no need to convert radicals or trigonometric functions to decimal form. For example, $\sqrt{2}$ and $\cos(50°)$ are acceptable answers. Nevertheless, simplify your answers when possible. For example, $4^{\frac{3}{2}} = 8$ and $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$.

- **Extra Paper.** There is a sheet of paper for scratch work at the end of the exam. If you need more space to write solutions, get more paper from the proctor. If you decide to use extra paper, please write that you have done so on the front of the exam and please staple the extra sheets to the rest of the exam.

1. (a) (12 points) Let f and g be functions of a single variable. Assume that f and g are twice differentiable. Show that $v(x,t) = f(x - ct)$ and $w(x,t) = g(x + ct)$ are solutions of the partial differential equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t). \tag{1}$$

Solution: We compute

$$\begin{aligned} v_t(x,t) &= -cf'(x - ct), \\ v_{tt}(x,t) &= c^2 f''(x - ct), \\ v_x(x,t) &= f'(x - ct), \text{ and} \\ v_{xx}(x,t) &= f''(x - ct). \end{aligned}$$

Therefore, we have

$$v_{tt}(x,t) = c^2 f''(x - ct) = c^2 v_{xx}(x,t).$$

A similar calculation shows that w also satisfies $w_{tt}(x,t) = c^2 w_{xx}(x,t)$.

Note: The partial differential equation (1) is called the wave equation, since it models waves on a string and other similar phenomena.

(b) (13 points) Find a function f which satisfies

$$\frac{\partial f}{\partial x} = 2x + y^2 \text{ and } \frac{\partial f}{\partial y} = 2xy. \tag{2}$$

Solution: First, we recognize that $f(x,y) = x^2 + xy^2$ satisfies $\frac{\partial f}{\partial x} = 2x + y^2$. We then check that $f(x,y)$ also satisfies $\frac{\partial f}{\partial y} = 2xy$. Thus, $f(x,y) = x^2 + xy^2$ is a function which satisfies (2).

2. (a) (13 points) Let $f(x,y) = x^2 + 2y^2 + e^{xy-2}$. Find the tangent line to the level curve $f(x,y) = 7$ at the point $(2,1)$.

Solution: The tangent line is normal to $\nabla f(2,1)$. We compute

$$\nabla f(x,y) = \left\langle 2x + ye^{xy-2}, 4y + xe^{xy-2} \right\rangle,$$

and so $\nabla f(2,1) = \langle 5,6 \rangle$. We can choose any vector orthogonal to $\nabla f(2,1) = \langle 5,6 \rangle$ as the direction vector of the tangent line; we observe that $\langle 6,-5 \rangle$ is orthogonal to $\langle 5,6 \rangle$ since we have $\langle 5,6 \rangle \cdot \langle 6,-5 \rangle = 0$. Thus,

$$\mathbf{r}(t) = \langle 2,1 \rangle + \langle 6,-5 \rangle\, t$$

is a vector parametrization for the tangent line.

Note: There are many correct ways to find and express the tangent line. For a complete discussion, see the solutions to the practice exam.

(b) (12 points) Draw a contour map of the function $f(x,y) = e^{xy}$. The contour map should include the three level curves $1 = f(x,y)$, $e = f(x,y)$, and $e^{-1} = f(x,y)$ in addition to one other curve of your choice. Be sure to label each curve with the value of f on that curve.

Solution: The level curves are defined by the equation

$$c = f(x,y) = e^{xy}.$$

Taking the logarithm of both sides and solving for y as a function of x, we see that the level curves for $c \neq 1$ are the hyperbolas

$$y = \frac{\ln(c)}{x}.$$

The level curve $1 = c = f(x,y)$ is special; it is described by the equation $\ln(1) = 0 = xy$. Since this equation is solved if and only if at least one of x or y is zero, the level curve $1 = f(x,y)$ consists of the x- and y-axes.

3. Let $\mathcal{S}$ be the surface defined by

$$z^2 + 4x^2 - y^2 - 16 = 0.$$

(a) (10 points) Find an equation for the tangent plane to $\mathcal{S}$ at the point $(0, 3, 5)$.

Solution: Let $g(x, y, z) = z^2 + 4x^2 - y^2 - 16$, and recall that $\nabla g(0, 3, 5)$ is normal to the tangent plane to the surface $g(x, y, z) = 0$ at the point $(0, 3, 5)$. We compute

$$\nabla g(x, y, z) = \langle 8x, -2y, 2z \rangle,$$

and so $\nabla g(0, 3, 5) = \langle 0, -6, 10 \rangle$. It follows that

$$-6(y - 3) + 10(z - 5) = 0$$

is an equation for the tangent plane.

(b) (15 points) Find a point (x, y, z) on $\mathcal{S}$ where the normal vector to the tangent plane to $\mathcal{S}$ is parallel with $\langle 8, -1, 1 \rangle$.

Solution: As above, the tangent plane to $\mathcal{S}$ at the point (x, y, z) is normal to the vector $\nabla g(x, y, z) = \langle 8x, -2y, 2z \rangle$. Thus, a point (x, y, z) on $\mathcal{S}$ where the tangent plane is normal to $\langle 8, -1, 1 \rangle$ satisfies the two equations

$$\langle 8x, -2y, 2z \rangle = \lambda \langle 8, -1, 1 \rangle, \text{ and}$$
$$z^2 + 4x^2 - y^2 - 16 = 0.$$

To solve the above system of equations, we use the first equation to express x, y, and z in terms of λ. We have

$$8x = 8\lambda, -2y = -\lambda, \text{ and } 2z = \lambda,$$

which gives

$$x = \lambda, y = \frac{\lambda}{2}, \text{ and } z = \frac{\lambda}{2}.$$

We now substitute these expressions in the equation $z^2 + 4x^2 - y^2 - 16 = 0$ to find

$$\left(\frac{\lambda}{2}\right)^2 + 4\lambda^2 - \left(\frac{\lambda}{2}\right)^2 - 16 = 0.$$

Thus, $\lambda = \pm 2$. We conclude that the points $(x, y, z) = \left(\lambda, \frac{\lambda}{2}, \frac{\lambda}{2}\right) = (\pm 2, \pm 1, \pm 1)$ lie on $\mathcal{S}$ and that the tangent plane to $\mathcal{S}$ at these points is normal to $\langle 8, -1, 1 \rangle$.

4. (a) (13 points) Let

$$f(x,y) = \frac{x^4 \sin(y)}{x^4 + y^2}.$$

Does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist? If so, find the limit.

Solution: We use the squeeze theorem to show that the limit is zero. First, observe that $0 \leq x^4 \leq x^4 + y^2$. This implies

$$0 \leq \frac{x^4}{x^4 + y^2} \leq 1.$$

Now $|\sin(y)| \geq 0$, so we can multiply the above inequality by $|\sin(y)|$ to find

$$0 \leq \frac{x^4 |\sin(y)|}{x^4 + y^2} \leq |\sin(y)|.$$

(*Note*: Remember that multiplying an inequality by a positive number preserves the inequality, while multiplying by a negative number reverses it. Multiplying by $\sin(y)$ does not lead to a correct inequality, since $\sin(y)$ assumes both positive and negative values.) Therefore,

$$-|\sin(y)| \leq \frac{x^4 \sin(y)}{x^4 + y^2} \leq |\sin(y)|.$$

Finally, we observe that

$$\lim_{(x,y)\to(0,0)} -|\sin(y)| = \lim_{(x,y)\to(0,0)} |\sin(y)| = 0.$$

Thus, by the squeeze theorem, $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

(b) (12 points) Let

$$h(x,y) = \frac{x^2y^3}{x^4+y^6}.$$

Does $\lim_{(x,y)\to(0,0)} h(x,y)$ exist? If so, find the limit.

Solution: We show that the limit does not exist by considering the limits along the paths

$$\mathbf{c}_1(t) = \langle t, 0 \rangle \text{ and } \mathbf{c}_2(t) = \left\langle t^3, t^2 \right\rangle.$$

Both paths are continuous, and they satisfy $\mathbf{c}_1(0) = \mathbf{c}_2(0) = (0,0)$. Thus, if $\lim_{(x,y)\to(0,0)} h(x,y)$ were to exist, we would have $\lim_{t\to 0} h(\mathbf{c}_1(t)) = \lim_{t\to 0} h(\mathbf{c}_2(t))$. However, we compute that

$$\lim_{t\to 0} h(\mathbf{c}_1(t)) = \lim_{t\to 0} \frac{t^2 0^3}{t^4+0^6} = \lim_{t\to 0} \frac{0}{t^4} = 0,$$

but

$$\lim_{t\to 0} h(\mathbf{c}_2(t)) = \lim_{t\to 0} \frac{(t^3)^2(t^2)^3}{(t^3)^4+(t^2)^6} = \lim_{t\to 0} \frac{t^{12}}{2t^{12}} = \frac{1}{2} \neq 0.$$

Therefore, $\lim_{(x,y)\to(0,0)} h(x,y)$ does not exist.

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