Math 32A Exam #1 Fall 2013

[redacted] Initial of Last Name: I

[redacted]iscussion Section: 3D

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- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers. If you use the back of a page please indicate that you have work on the reverse side.
- You may not use books, notes, calculators, mobile phones, or any outside help during this exam. You may not collaborate with other students in any way during this exam.

Problem	Points	Score
1	20	20
2	20	19
3	10	10
4	10	10
5	10	9
Total	70	68

1. (20 points) For each of the parts, (a)–(e), indicate whether the statement is true or false.

(a) (4 points) For two nonzero vectors $\mathbf{u}, \mathbf{v}$, $||\text{Proj}_{\mathbf{u}}\mathbf{v}|| \le ||\mathbf{v}||$.

$\boxed{\text{True}}$ $||\text{Proj}_u \vec{v}|| = ||\vec{v}|| \cos\theta \le ||\vec{v}||$

(b) (4 points) If $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = 0$, then $||\mathbf{u}|| = ||\mathbf{v}||$.

$\boxed{\text{True}}$ ∵ dot product is distributive

$u^2 - v^2 = 0 \Rightarrow u^2 = v^2 \Rightarrow ||\vec{u}|| = ||\vec{v}||$

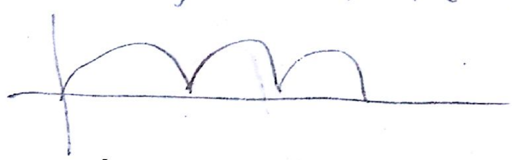
(c) (4 points) If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

$\boxed{\text{False}}$ $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} \Rightarrow \vec{a} \times (\vec{b} - \vec{c}) = \vec{0}$

As long as $\vec{b} - \vec{c} \perp \vec{a}$, this is true

(d) (4 points) The vector-valued function $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t \rangle$ is the cycloid. You know that the graph of $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ is a function $y = f(x)$ in the x, y-plane. Since $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$ exists for all $t \in \mathbb{R}$, the derivative $\frac{df}{dx}$ must also exist for *all* values of x.

$\boxed{\text{False}}$ consider $t = 0$, so $x(0)$, $f'(0)$ clearly D.N.E



(e) (4 points) For any plane in $\mathbb{R}^3$ and any line not intersecting the plane, there is a unique point on the line which is the closest point to the plane.

$\boxed{\text{False}}$ If the line does not intersect the plane, the line and the plane must be parallel.

∴ Any point on the line is equidistant from the plane.

20

9 4 1

19

2. (20 points) This question has parts (a)–(d). Let $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 1\mathbf{k}$ and $\mathbf{w} = \mathbf{i} - \mathbf{j}$.

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(a) (5 points) Find the unit vector in the same direction as $\mathbf{v}$.

$$\hat{v} = \frac{1}{\|\vec{v}\|}\vec{v}$$

$$= \frac{1}{\sqrt{3^2+2^2+1^2}}\langle 3, 2, 1\rangle$$

$$= \frac{1}{\sqrt{14}}\langle 3, 2, 1\rangle$$

$$= \boxed{\left\langle \frac{3}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}} \right\rangle}$$ ✓

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(b) (5 points) Find the angle between $\mathbf{v}$ and $\mathbf{w}$.

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$$ ✓

$$(3)(1)+(2)(-1)+(1)(0) = \sqrt{3^2+2^2+1^2}\sqrt{1^2+1^2}\cos\theta$$

$$1 = \sqrt{\cancel{14}\ 28}\cos\theta$$

$$\cos\theta = \frac{\sqrt{34}}{34}$$

$$\boxed{\theta = \arccos\frac{\sqrt{\cancel{34}}}{\cancel{34}}}$$

$\frac{\sqrt{28}}{28}$

(c) (5 points) Compute the projection of **w** onto **v**.

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$$\text{Proj}_{\vec{v}}\vec{w} = \frac{\vec{w}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

$$= \frac{1}{14}\langle 3, 2, 1\rangle$$

$$= \boxed{\left\langle \frac{3}{14}, \frac{1}{7}, \frac{1}{14} \right\rangle}$$

(d) (5 points) Find an equation for the plane which contains **v**, **w**, and **0** (the zero-vector).

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$\vec{N}$ must be normal to both $\vec{w}$ and $\vec{v}$.

$\therefore$ one possible $\vec{N}$ is

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 1 \\ 1 & -1 & 0 \end{vmatrix} = [(2)(0) - (-1)(1)]\hat{i} - [(3)(0) - (1)(1)]\hat{j} + [(3)(-1) - (2)(1)]\hat{k}$$

$$= \hat{i} + \hat{j} - 5\hat{k} = \langle 1, 1, -5 \rangle$$

$\therefore$ The plane is

$$\vec{N}\cdot(\vec{r} - \vec{0}) = 0$$

$$\boxed{x + y - 5z = 0}$$

3. (10 points) Find a parameterization for the intersection of the following two surfaces in $\mathbb{R}^3$.

$$x^2 + y^2 = 9 \quad \text{and} \quad z = x + y$$

For your parameterization, make sure to state its domain.

We can parametrize the first surface by

$x = 3\cos t$, $y = 3\sin t$, $z = z$, $t \in [0, 2\pi]$ because $x^2+y^2 = (3\cos t)^2 + (3\sin t)^2 = 9(\cos^2 t + \sin^2 t) = 9$

Plugging this into the second equation, we find

$z = 3\cos t + 3\sin t$.

$\therefore$ The intersection is parametrized by the vector

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

$\boxed{\vec{r}(t) = \langle 3\cos t, 3\sin t, 3\cos t + 3\sin t \rangle, \quad t \in [0, 2\pi]}$ ✓

4. (10 points) This problem has parts (a) and (b). Consider the following four points in $\mathbb{R}^3$: $(0,0,0), (1,1,0), (0,0,1), (1,1,2)$

(a) (5 points) Show that these four points are contained in a single plane.

let $\vec{A} = \langle 0,0,0\rangle$, $\vec{B} = \langle 1,1,0\rangle$, $\vec{C} = \langle 0,0,1\rangle$, $\vec{D} = \langle 1,1,2\rangle$

we first find the plane containing $\vec{A}$, $\vec{B}$, and $\vec{C}$.

$\vec{AB} = \langle 1,1,0\rangle$, $\vec{AC} = \langle 0,0,1\rangle$

$\therefore \vec{N} = \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 1,-1,0\rangle$ ✓

The plane equation is

$\vec{N}\cdot(\vec{r}-\vec{A}) = 0 \Rightarrow x - y = 0.$

Since $\vec{D} = \langle 1,1,2\rangle$, and $(1)-(1) = 0$.

$\vec{D}$ satisfies the equation of the plane.

$\therefore$ $\vec{D}$ is contained in the plane containing $\vec{A}$, $\vec{B}$, and $\vec{C}$.

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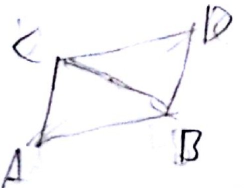

(b) (5 points) Find the area of the quadrilateral that these four points define by using the methods we have learned in Math 32A.

Note $\vec{AB} = \langle 1,1,0\rangle$, $\vec{AC} = \langle 0,0,1\rangle$, $\vec{BD} = \langle 0,0,2\rangle$, $\vec{CD} = \langle 1,1,1\rangle$

$$A_{ADC} = \frac{1}{2}\|\vec{AB}\times\vec{AC}\| = \frac{1}{2}\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right\| = \frac{1}{2}\|\langle 1,-1,0\rangle\| = \frac{\sqrt{2}}{2}$$

we used $\vec{BD}$ and $\vec{CD}$ here instead but that doesn't matter as we will take the magnitude anyway

$$A_{BCD} = \frac{1}{2}\|\vec{DC}\times\vec{DB}\| = \frac{1}{2}\left\|\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix}\right\| = \frac{1}{2}\|\langle -2,2,0\rangle\| = \sqrt{2}$$

$$A_{ABCD} = A_{ABC} + A_{BCD}$$

$$= \frac{\sqrt{2}}{2} + \sqrt{2} = \boxed{\frac{3}{2}\sqrt{2}\text{ unit}^2}$$ ✓

5. (10 points) Let $\mathbf{r}_1(t) = \langle t^2, t^3, t\rangle$ and $\mathbf{r}_2(s) = \langle s-2, 3s-8, s-2\rangle$

(a) (5 points) Find the point of intersection between $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$.

$\vec{r}_1(t) = \vec{r}_2(s)$

$\langle t^2, t^3, t\rangle = \langle s-2, 3s-8, s-2\rangle$

$x = t^2 = s-2 \quad (1)$

$y = t^3 = 3s-8 \quad (2)$

$z = t = s-2 \quad (3)$

Substitute (3) into (1), we find

$(s-2)^2 = s-2 \Rightarrow (s-2)^2-(s-2)=0 \Rightarrow (s-2)(s-2-1)=0 \Rightarrow (s-2)(s-3)=0$

$\therefore s=2$ or $s=3$

The corresponding t's are $t=0$ and $t=1$

Plugging them into (2)

For $s=2$, $t=0$, we have $(0)^3 = 3(2)-8 \Rightarrow 0 = -2$, a contradiction

For $s=3$, $t=1$, we have $(1)^3 = 3(3)-8 \Rightarrow 1 = 1$ which is true

$\therefore$ The intersection occurs at $\vec{r}_1(1)$ or $\vec{r}_2(3)$

$\vec{r}_1(1) = \langle 1^2, 1^3, 1\rangle = \boxed{\langle 1, 1, 1\rangle}$

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(b) (5 points) Determine $\cos\theta$, where θ is the angle between $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ at the point where they intersect.

$\vec{r}_1'(t) = \langle 2t, 3t^2, 1\rangle \quad \vec{r}_2'(s) = \langle 1, 3, 1\rangle$

$\vec{r}_1'(1) = \langle 2(1), 3(1)^2, 1\rangle \quad \vec{r}_2'(3) = \langle 1, 3, 1\rangle$

$= \langle 2, 3, 1\rangle$

$\vec{r}_1'(1)\cdot\vec{r}_2'(3) = \|\vec{r}_1'(1)\|\,\|\vec{r}_2'(3)\|\cos\theta$

$(2)(1)+(3)(3)+(1)(1) = \sqrt{(1)^2+3^2+1^2}\cdot\sqrt{2^2+3^2+1^2}\cos\theta$

$12 = \cancel{\sqrt{12}}\ \sqrt{14}\ \sqrt{11}\cos\theta$

$\cos\theta = \frac{12}{\sqrt{12}\sqrt{11}} = \frac{\sqrt{12}}{\sqrt{11}} = \boxed{\frac{\sqrt{132}}{11}}$

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