Formulas

- General Curvature Formulas:

$$\kappa(s) = \left\| \frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s} \right\|$$
$$\kappa(t) = \frac{1}{v(t)} \|\mathbf{T}'(t)\|$$
$$= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$

- Curvature of a Graph $y = f(x)$ in $\mathbb{R}^2$:

$$\kappa(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$$

- Decomposition of the Acceleration Vector:

$$\mathbf{a}(t) = a_{\mathbf{T}}(t)\mathbf{T}(t) + a_{\mathbf{N}}(t)\mathbf{N}(t)$$
$$a_{\mathbf{T}}(t) = v'(t)$$
$$a_{\mathbf{N}}(t) = \kappa(t)v(t)^2$$

- Trigonometric Limits:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$
$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

1. (30 points) Find the Frenet frame (**T**, **N**, and **B**) to the curve parametrized by

$$\mathbf{r}(t) = \left\langle 2\sqrt{2}e^{t/2}, \frac{1}{2}e^{2t}, -e^{-t} \right\rangle$$

at $t = 0$. **Hint:** It might help to be on the lookout for perfect squares!

$r' = \langle \sqrt{2}e^{t/2}, e^{2t}, e^{-t} \rangle$

$\sqrt{2e^{t} + e^{4t} + e^{-2t}}$

$\sqrt{(e^{2t} + e^{-t})^2}$

$e^{2t} + e^{-t}$

$T = \dfrac{\langle \sqrt{2}e^{t/2}, e^{2t}, e^{-t} \rangle}{\sqrt{2e^{t} + e^{4t} + e^{-2t}}}$

$T(0) = \dfrac{\langle \sqrt{2}, 1, 1 \rangle}{2}$

$T' = \dfrac{\langle \frac{\sqrt{2}}{2}e^{t/2}, 2e^{2t}, -e^{-t} \rangle (e^{2t} + e^{-t}) - (2e^{2t} - e^{-t})(\langle \sqrt{2}e^{t/2}, e^{t}, e^{-t} \rangle}{(e^{2t} + e^{-t})^2}$

$T'(0) = \dfrac{2\langle \frac{\sqrt{2}}{2}, 2, -1 \rangle - \langle \sqrt{2}, 1, 1 \rangle}{4}$

$T'(0) = \dfrac{\langle 0, 3, -3 \rangle}{4} = \langle 0, \frac{3}{4}, -\frac{3}{4} \rangle$

$N = \dfrac{\langle 0, \frac{3}{4}, -\frac{3}{4} \rangle}{\sqrt{\frac{9}{16} + \frac{9}{16}}} = \dfrac{\langle 0, \frac{3}{4}, -\frac{3}{4} \rangle}{\frac{3\sqrt{2}}{4}} = \dfrac{\langle 0, 1, -1 \rangle}{\sqrt{2}}$

$B = T \times N = \frac{1}{2\sqrt{2}} \langle 1(-1) - (1)(1), (1)(0) - \sqrt{2}(-1), \sqrt{2}(1) - (1)(0) \rangle$

$B = \frac{1}{2\sqrt{2}} \langle -2, \sqrt{2}, \sqrt{2} \rangle$

$B = \langle -\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2} \rangle$

2. (25 points) Suppose that a particle travels along a path parametrized by the vector-valued function $\mathbf{r}(t)$, and its acceleration vector at time t is given by

capital Letters are constants

$$\mathbf{a}(t) = \langle 2, 6t, -\cos t + e^t \rangle$$

Suppose moreover that its initial position and velocity are given by

$$\mathbf{r}(0) = \langle -5, 1, 3 \rangle$$
$$\mathbf{v}(0) = \langle 0, 1, 3 \rangle$$

Find the parametrization $\mathbf{r}(t)$ of the particle's path.

$V = \int a = \langle 2t + A, 3t^2 + B, -\sin t + e^t + C \rangle$

$v(0) = \langle 0, 1, 3 \rangle = \langle 2(0) + A, 3(0)^2 + B, -\sin(0) + e^0 + C$

$\langle 0, 1, 3 \rangle = \langle A, B, 1 + C \rangle$

$V = \langle 2t, 3t^2 + 1, -\sin t + e^t + 2 \rangle$

$r = \int v = \langle t^2 + D, t^3 + t + E, \cos t + e^t + 2t + F \rangle$

$r(0) = \langle -5, 1, 3 \rangle = \langle 0 + D, 0 + 0 + E, 1 + 1 + 0 + F \rangle$

$\langle -5, 1, 3 \rangle = \langle D, E, 2 + F \rangle$

$r = \langle t^2 - 5, t^3 + t + 1, \cos t + e^t + 2t + 1 \rangle$ ✓

3. (25 points) Calculate all second partial derivatives of the function

$$f(x,y) = \sin(x^2)\ln(x+y)$$

(which is well-defined when $x + y > 0$) and show that the conclusion of Clairaut's Theorem holds for this function.

$$f_x = [2x\cos x^2](\ln(x+y)) + \frac{1}{x+y}(\sin x^2)$$

$$f_y = \frac{\sin x^2}{x+y}$$

$$f_{yx} = \frac{2x\cos x^2(x+y) - \sin x^2}{(x+y)^2}$$

$$f_{xy} = \frac{2x\cos^2}{x+y} + \sin x^2\left(-\frac{1}{(x+y)^2}\right)$$

$$f_{xy} = \frac{2x\cos^2(x+y) - \sin x^2}{(x+y)^2}$$

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$f_{yx} = f_{xy}$ so theorem holds

$$f_{yy} = \frac{-\sin x^2}{(x+y)^2}$$

$$f_{xx} = [2\cos x^2 - 4x^2\sin x^2]\ln(x+y) + \frac{2x\cos x^2}{x+y} + \left[\frac{2x\cos x^2(x+y) - \sin x^2}{(x+y)^2}\right]$$

4. Compute the following limits or show that they do not exist:

$\sin^6(x) \geq 0$

(a) (10 points)

$$f = \lim_{(x,y)\to(0,0)} \frac{x^3 y}{\sin^6(x) + y^2}$$

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$0 \leq |f| \leq \left|\frac{x^3 y}{y^2}\right|$

$0 \leq |f| \leq \left|\frac{r^3 \cos^3\theta \, r\sin\theta}{r^2 \sin^2\theta}\right|$ $r^2 = x^2+y^2$

$0 \leq |f| \leq |r^2 \cot\theta \cos^2\theta|$ $\cot\theta \leq 1$ & $\cos^2\theta \leq 1$

no!

$0 \leq |f| \leq \lim_{(x,y)\to(0,0)} x^2+y^2$

So by the squeese theorem

$0 \leq \lim_{x,y\to 0} |f| \leq 0$

$\lim_{(x,y)\to(0,0)} |f| = 0 \therefore \lim_{(x,y)\to(0,0)} f = 0$

the originul limit DNE

(b) (10 points)

$$\lim_{(x,y)\to(0,0)} \frac{1 - 2\cos\left(\sqrt{x^2+y^2}\right) + \cos^2\left(\sqrt{x^2+y^2}\right)}{x^2+y^2} \cdot \sin\left(\ln\left(\frac{1}{x^2+y^4}\right)\right)$$

$\lim_{(x,y)\to(0,0)} \frac{1-2\cos(\sqrt{x^2+y^2}) + \cos^2(\sqrt{x^2+y^2})}{x^2+y^2} \cdot \lim_{(x,y)\to(0,0)} \sin\left(\ln\left(\frac{1}{x^2+y^4}\right)\right)$

$\lim_{x\to 0} \frac{1-2\cos(\sqrt{x^2}) + \cos^2(\sqrt{x^2})}{x^2} \cdot \lim_{x\to 0} \sin\left(\ln\left(\frac{1}{x^2}\right)\right)$

0

$\frac{1-2+1}{0} \cdot \sin\left(\ln\left(\frac{1}{0}\right)\right) = \text{undefined}$

$y = mx$

$\lim_{x\to 0} \frac{1-2\cos(x\sqrt{1+m^2}) + \cos^2(x\sqrt{1+m^2})}{2mx^2} \cdot \lim_{x\to 0} \sin\left(\ln\left(\frac{1}{x^2+m^4x^4}\right)\right)$

$\lim_{x\to 0}$