Midterm 2

Name:

UID:

Section: Tuesday: Thursday:

Tuesday	Thursday	TA
1A	1B	TA: Yurun Ge
1C	1D	TA: Benjamin Johnsrude
1E	1F	TA: Tianqi Wu

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. You may not use calculators, books, notes, or any other material to help you. Please make sure your phone is silenced and stowed where you cannot see it. You may use the front and back of the page for your answers. **Do not write answers for one question on the page of another question.** If you need scratch paper, please ask one of the proctors. You must **show all your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	40

UID:

1. Consider the function $f(x, y) = \sin x + y^2 e^x$.

   (a) (6 points) Find an equation of the tangent plane to the graph of f at $(0, 3, f(0, 3))$.

   (b) (4 points) Estimate the value of $f(0.1, 2.9)$ using linear approximation.

a. $f(0,3) = \sin 0 + 3^2 e^0 = 9 \quad (0,3,9)$

$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f_x = \cos x + y^2 e^x \quad f_x(0,3) = \cos 0 + 3^2 e^0 = 10$

$(1) + 9(1)$

$z = 9 + 10(x-0) + 6(y-3)$

$z = 9 + 10x + 6y - 18$

$f_y = 2e^x y \quad f_y(0,3) = 2e^0(3) = 6$

$(2)(1)(3)$

$\boxed{z = 10x + 6y - 9}$

b. $L(x,y) = 10x + 6y - 9$

$f(x,y) + f_x \Delta x + f_y \Delta y$

$9 + 10(.1) + 6(-.1)$

$9 + 1 \quad -.6 = 9.4$

$L(.1, 2.9) = 10(.1) + 6(2.9) - 9$

$= 1 + 17.4 - 9$

$= \boxed{9.4}$

111'1001 11 01
11'1101 1 10
11

2. (a) (5 points) Consider the function $f(x,y)$ defined for $(x,y) \neq (0,0)$ by

$$f(x,y) = x \sin\left(\frac{xy}{x^4+y^4}\right).$$

Show that one can assign a value to $f(0,0)$ to make this function continuous.

(b) (5 points) Evaluate the following limit or show it doesn't exist:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^4}{x^4+y^8}.$$

$\left|\sin \frac{xy}{x^4+y^4}\right| \le 1$

a. $f(0,0) = \lim_{(x,y)\to(0,0)} x \sin\left(\frac{xy}{x^4+y^4}\right)$

along $x=0$ y-axis, $\lim_{y\to 0} 0 \cdot \sin\left(\frac{0}{y^4}\right) = 0$

along $y=0$ x-axis, $\lim_{x\to 0} x \cdot \sin\left(\frac{0}{x^4}\right) = \lim_{x\to 0} x \cdot 0 = 0$

$0 \le \left|x\right|\left|\sin\frac{xy}{x^4+y^4}\right| \le |x|$

$0 \le \left|\lim_{(x,y)\to(0,0)} x \sin\frac{xy}{x^4+y^4}\right| \le \left|\lim_{x\to 0} x\right| = 0$

$0 \le \lim_{(x,y)\to(0,0)} x \sin\frac{xy}{x^4+y^4} \le 0$

$\therefore$ Squeeze Theorem shows $\lim_{(x,y)\to(0,0)} x\sin\left(\frac{xy}{x^4+y^4}\right) = 0.$

$\therefore f(0,0) = 0$ meaning function is continuous.

b. $\lim_{(x,y)\to(0,0)} \frac{x^2y^4}{x^4+y^8}$

along $x=0$ y-axis: $\lim_{y\to 0} \frac{0}{y^8} = 0$

along $y=0$ x-axis: $\lim_{x\to 0} \frac{0}{x^4} = 0$

along $x=y$ $\lim_{x\to 0} \frac{x^2x^4}{x^4+x^4} = \lim_{x\to 0} \frac{x^6}{2x^4}$

$= \lim_{x\to 0} \frac{x^2}{2} = 0$

along $y = x^{1/2}$ $\lim_{x\to 0} \frac{x^2(x^{1/2})^4}{x^4+(x^{1/2})^8} = \lim_{x\to 0} \frac{x^2(x^2)}{x^4+x^4}$

$= \lim_{x\to 0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$

Since $0 \neq \frac{1}{2}$,

$\lim_{(x,y)\to(0,0)} \frac{x^2y^4}{y^4+y^8}$ DNE

Does Not exist.

3. (a) (3 points) In the xy-plane, sketch the contour diagram of $f(x,y) = x^2 - y^2$ containing the level curves corresponding to $c = -1, 0, 1, 2$.

(b) (3 points) Calculate $\nabla f(x,y)$. Draw $\nabla f(0,1)$ based at the point $(0,1)$ in the contour diagram of part (a).

(c) (1 point) State Clairaut's Theorem on higher order partial derivatives.

(d) (3 points) Consider the function $f(x,y,z) = x^3 e^y \sin z + \frac{z^2 e^{2y}}{2+\cos z}$. Calculate $\frac{\partial^3 f}{\partial x \partial z \partial y}$.

$c = x^2 - y^2$

$c=1,\ 1 = x^2 - y^2$

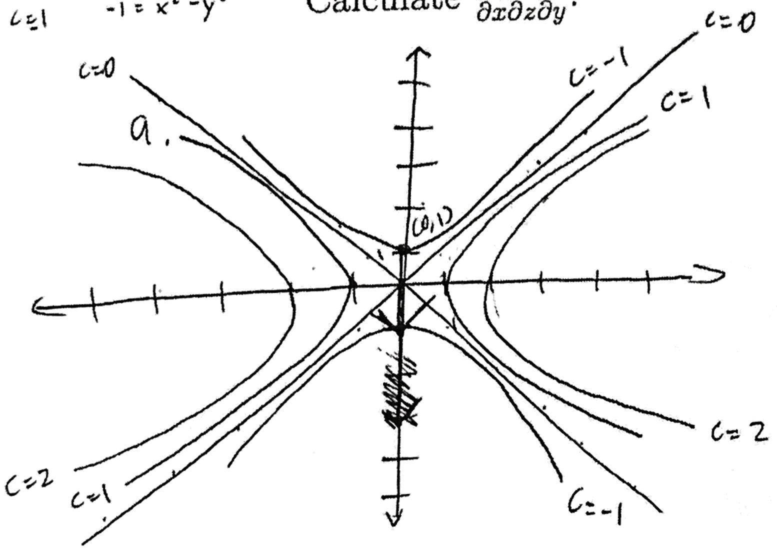
$c=0 = 0 = x^2 - y^2$

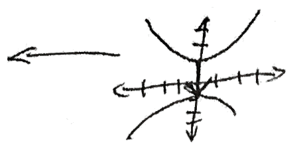
$x^2 = y^2$

$c=-1\ \ -1 = x^2 - y^2$

b. $\boxed{\nabla f = \langle 2x, -2y \rangle} = \nabla f(x,y)$

$\nabla f(0,1) = \langle 0, -2 \rangle$ at $P(0,1)$





c. If second order partial derivative both exists and are continuous, $f_{xy} = f_{yx}$ for this mixed partial. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ can be solve either way, any order.

In the case of 3a,

$f_x = 2x \Rightarrow f_{xy} = 0$

$f_y = -2y \Rightarrow f_{yx} = 0$

d. $f(x,y,z)$

$f_x = 3x^2 e^y \sin z$

$f_{xz} = 3x^2 e^y \cos z$

$f_{xzy} = \boxed{3x^2 e^y \cos z} = \frac{\partial^3 f}{\partial x \partial z \partial y}$

$c > 0$, one sheet

$c < 0$, two

a. no speed change, just curvature

c. no curvature, just speed change

4. (a) (3 points) Suppose $\mathbf{r}(t) = \langle 3\sin t, 3\cos t, 4t\rangle$. Find the acceleration vector $\mathbf{a}(t)$.

(b) (4 points) For $\mathbf{r}(t)$ as in (a), calculate the unit tangent and unit normal vector at $t = 0$ and find the tangential component and normal component of $\mathbf{a}(t)$ at $t = 0$.

(c) (3 points) A car is moving along a path $\mathbf{r}(t)$ which satisfies $\mathbf{r}(0) = \langle 1, 2, 0\rangle$ and $\mathbf{r}'(0) = \langle 2, 3, -6\rangle$. At time $t = 0$ the unit vector $\mathbf{N} = \langle 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$ is normal to the curve. Suppose that at time $t = 0$, the car is slowing down at a rate of 2m/s^2 and is not changing direction. Find the acceleration vector at time $t = 0$.

$a = a_T T + a_N N$

$a_T T = \left(\frac{a\cdot v}{v\cdot v}\right) v$

$T = \frac{v}{\|v\|}$ $\quad a_T = a\cdot T = \frac{a\cdot v}{\|v\|}$

$a_N = \sqrt{\|a\|^2 - \|a_T\|^2} = \frac{\|a \times v\|}{\|v\|}$

$\|a_N N\| = a_N$

$a_N N = a - a_T \cdot T$

$\langle \frac{3}{5}, 0, \frac{4}{5}\rangle$

a. $r(t) = \langle 3\sin t, 3\cos t, 4t\rangle$

$r'(0) = \langle 3, 0, 4\rangle$ $\quad v(t) = r'(t) = \langle 3\cos t, -3\sin t, 4\rangle$

$r''(0) = \langle 0, -3, 0\rangle$ $\quad \boxed{a(t) = r''(t) = \langle -3\sin t, -3\cos t, 0\rangle}$

$a_N = \sqrt{\|a\|^2 - \|a_T\|^2} = \sqrt{9\sin^2 t + 9\cos^2 t + 0 - 0}$
$= \sqrt{9} = 3$

$a = a_T T + a_N N$, $\quad a_T = 0 \quad a_N = 3$

$\boxed{a = 3N}$ where $N = \langle -\sin t, -\cos t, 0\rangle$

$\|v\| = \|r'(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = 5$

$T = \frac{\langle 3\cos t, -3\sin t, 4\rangle}{5} = \langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5}\rangle$

$a_T = a\cdot T = \langle -3\sin t, -3\cos t, 0\rangle \cdot \langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5}\rangle$
$= -\frac{9}{5}\sin t\cos t + \frac{9}{5}\sin t\cos t + 0 = 0$

$a_T = \frac{a\cdot v}{\|v\|} = \frac{-9\cos t\sin t + 9\cos t\sin t + 0}{5} = 0$

$a_T T = \left(\frac{a\cdot v}{v\cdot v}\right) v = \left(\frac{0}{25}\right)\langle 3\cos t, -3\sin t, 4\rangle = 0$

$\frac{a_N N}{a_N} = \frac{\langle -3\sin t, -3\cos t, 0\rangle - 0}{3} = \langle -\sin t, -\cos t, 0\rangle$

$a(0) = \langle 0, -1, 0\rangle$

b. $T = \frac{v}{\|v\|}$, $\|v\| = 5$, $v(0) = \langle 3, 0, 4\rangle$

$= \frac{\langle 3, 0, 4\rangle}{5} = \boxed{\langle \frac{3}{5}, 0, \frac{4}{5}\rangle}$ unit tangent vector

$a_N N = a - a_T T$
$= \langle -3\sin t, -3\cos t, 0\rangle - \frac{0}{25}\langle 3\cos t, -3\sin t, 4\rangle$
$= \langle -3\sin t, -3\cos t, 0\rangle$

$a_N = \|a_N N\| = \sqrt{9} = 3$

$\frac{a_N N}{a_N} = \frac{\langle -3\sin t, -3\cos t, 0\rangle}{3}$

$N(t) = \langle -\sin t, -\cos t, 0\rangle$

$N(0) = \boxed{\langle 0, -1, 0\rangle}$
$t=0$

unit normal vector

$a = a_T T + a_N N$

$a = 0T + 3N \Rightarrow a = 3N$

c. $N(0) = \langle 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$ $\quad a_N = 0$ at $t = 0$

$a = a_T T + a_N N$ $\quad a_T = -2 \quad t = 0$

$T = \frac{r'(0)}{\|r'(0)\|} = \frac{\langle 2, 3, -6\rangle}{\sqrt{4+9+36}} = \frac{\langle 2, 3, -6\rangle}{7} = \langle \frac{2}{7}, \frac{3}{7}, -\frac{6}{7}\rangle$

$a = -2\langle \frac{2}{7}, \frac{3}{7}, -\frac{6}{7}\rangle + 0\langle 0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\rangle$

$a(0) = a_{t=0} = \boxed{\langle -\frac{4}{7}, -\frac{6}{7}, \frac{12}{7}\rangle}$

a_N curvature

a_T speed

$a_N N =$