Problem 1

(a) (1 point) 1 point if you circled your TA and discussion session on the previous page!

(b) (3 points) Let $f(x, y, z)$ be a function of three variables. Finish the definition: $\frac{\partial f}{\partial z}(a, b, c) = \ldots$

$$= \lim_{h \to 0} \frac{f(a,b,c+h) - f(a,b,c)}{h}$$

(c) (3 points) Find a parametrization of the curve in $\mathbb{R}^2$ given by the equation $2x^2 + y^2 - 4x + 2y + 2 = 0$ (Hint: classify it first).

$$2(x-1)^2 + (y+1)^2 = 1$$

This is ellipse $\frac{x^2}{(1/\sqrt{2})^2} + \frac{y^2}{1} = 1$ → $x = \frac{1}{\sqrt{2}} \cos t$, $y = \sin t$, $0 \leq t \leq 2\pi$

but shifted by 1 in positive x direction and by -1 in the y direction

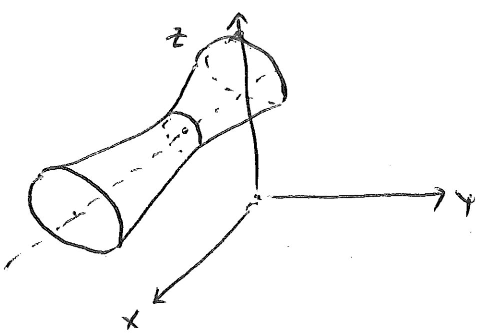
So $\begin{cases} x = 1 + \frac{1}{\sqrt{2}} \cos t \\ y = -1 + \sin t \end{cases}$, $0 \leq t \leq 2\pi$

(d) (3 points) (i) Classify, (ii) sketch, and (iii) find the axis of symmetry for the surface

$$(x-1)^2 - (y+3)^2 - (z-2013)^2 = -1$$

Multiply by -1: $-(x-1)^2 + (y+3)^2 + (z-2013)^2 = 1$

So it's hyperboloid of one sheet with axis of symmetry parallel to x-axis (shifted from (0,0,0) to (1,-3,2013))

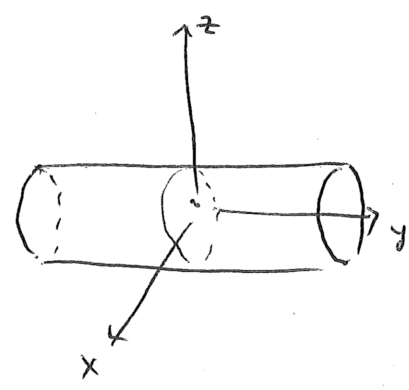


Axis of symmetry is

$$\begin{cases} x = 1 + t \\ y = -3 \\ z = 2013 \end{cases}$$

(e) (3 points) For the surface $x^2 + 5z^2 = 13$ in $\mathbb{R}^3$ classify all the traces parallel to the xy coordinate plane (mark all possibilities):

(i) Ellipses
(ii) Parabolas
(iii) Hyperbolas
(iv) Two intersecting lines
(v) Two parallel lines
(v) One line
(vi) A point
(vii) Empty



Problem 2

Let $\vec{r}(t) = \frac{3}{2}\sin(2t)\vec{\imath} - 2\sin(2t)\vec{\jmath} - \frac{5}{2}\cos(2t)\vec{k}$.

(a) (2 points) Compute the unit tangent vector $\vec{T}(t)$ at time t.

(b) (2 points) Compute the principal unit normal vector $\vec{N}(t)$ at time t.

(c) (2 points) Compute curvature $\kappa(0)$ at time $t = 0$.

(d) (2 points) Find the center of curvature to this trajectory at time $t = 0$.

(d) (2 points) Find the equation of the tangent line to this trajectory at time $t = 0$.

(e) (2 points) Find the equation of the osculating plane to this trajectory at time $t = 0$.

The promised curvature formulas:

$$\kappa(t) = \left\|\frac{d\vec{T}}{ds}\right\|$$

$$\kappa(t) = \frac{\|\vec{T}'(t)\|}{\|\vec{r}'(t)\|}$$

$$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

$$\kappa(x) = \frac{|f''(x)|}{(1+f'(x)^2)^{3/2}}$$

(a) $\vec{r}(t) = \left\langle \frac{3}{2}\sin 2t, -2\sin 2t, -\frac{5}{2}\cos 2t \right\rangle$

$\vec{r}'(t) = \langle 3\cos 2t, -4\cos 2t, 5\sin 2t \rangle$

$\|\vec{r}'(t)\| = \sqrt{9\cos^2 2t + 16\cos^2 2t + 25\sin^2 2t} = \sqrt{25} = 5$

So $\vec{T}(t) = \left\langle \frac{3}{5}\cos 2t, -\frac{4}{5}\cos 2t, \sin 2t \right\rangle$

(b) $\vec{T}'(t) = \left\langle -\frac{6}{5}\sin 2t, \frac{8}{5}\sin 2t, 2\cos 2t \right\rangle$

$\|\vec{T}'(t)\| = \sqrt{\frac{36}{25}\sin^2 2t + \frac{64}{25}\sin^2 2t + 4\cos^2 2t} = \sqrt{4} = 2$

$\vec{N}(t) = \frac{\vec{T}'(t)}{\|\vec{T}'(t)\|} = \left\langle -\frac{3}{5}\sin 2t, \frac{4}{5}\sin 2t, \cos 2t \right\rangle$

(c) $K(0) = \frac{\|\vec{T}'(0)\|}{\|\vec{r}'(0)\|} = \frac{2}{5}$

(d) Center of curvature is $\vec{r}(0) + \frac{1}{K(0)}\vec{N}(0) = \langle 0, 0, -\frac{5}{2}\rangle +$

$+ \frac{5}{2}\langle 0,0,1\rangle = \langle 0,0,0\rangle$, i.e. the origin

(d) Tangent line at $t=0$ is passing through $\vec{r}(0)$ with directional vector $\vec{r}'(0)$, so $\langle x,y,z\rangle = \langle 0,0,-5/2\rangle + \langle 3,-4,0\rangle \cdot t$

(e) Osculating plane must pass through $\vec{r}(0)$, i.e. $(0,0,-\frac{5}{2})$

with normal vector being $\vec{B}(0) = \vec{T}(0) \times \vec{N}(0) = \left\langle \frac{3}{5}, -\frac{4}{5}, 0\right\rangle \times \langle 0,0,1\rangle$

$$= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3/5 & -4/5 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\frac{4}{5}\vec{\imath} - \frac{3}{5}\vec{\jmath}$$

So plane is $-\frac{4}{5}x - \frac{3}{5}y = 0$

Problem 3

Let $f(x,y) = e^{x^2-y^2}$

(a) (2 points) Compute f_x and f_y.

(b) (2 points) Is f differentiable at $(1,-1)$? Justify your answer.

(c) (2 points) Find the differential of f at the point $(1,-1)$.

(d) (2 points) Find the equation of the tangent plane to the surface $z = f(x,y)$ at the point $(1,-1,1)$.

(e) (3 points) Suppose the point (x,y) changes from $(1,-1)$ to $(0.98,-0.99)$. Use linear approximation to estimate the change in the height of the surface $f(0.98,-0.99) - f(1,-1)$.

(a) $f_x(x,y) = 2x\, e^{x^2-y^2}$

$f_y(x,y) = -2y\, e^{x^2-y^2}$

(b) Note that $2x\, e^{x^2-y^2}$ is continuous everywhere:

$2x$ is polynomial $\Rightarrow$ continuous everywhere

$e^{x^2-y^2}$ is composition of ~~[illegible]~~ x^2-y^2 and e^t. Both of them are continuous everywhere $\Rightarrow$ their composition is too

Thus $2x\, e^{x^2-y^2}$ is continuous everywhere as ~~[illegible]~~ a product of two continuous functions

Similarly, $-2y\, e^{x^2-y^2}$ is continuous everywhere

By a theorem in class, f is differentiable everywhere

(c) $f_x(1,-1) = 2$

$f_y(1,-1) = 2$

So $df = 2\,dx + 2\,dy$

(d) Tangent plane is $z = 1 + 2(x-1) + 2(y+1)$

i.e. $z = 2x + 2y + 1$

(e) $\Delta z \approx dz = 2\,dx + 2\,dy = 2\cdot(-0.02) + 2\cdot(0.01) = -0.02$

Problem 4

(a) (5 points) Compute the following limit or prove that it does not exist. Justify all the steps.

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

Take path $x=0$: $f=0 \longrightarrow 0$

Take path $x=y^3$: $f=\frac{y^3y^3}{y^6+y^6}=\frac{1}{2}\xrightarrow[y\to 0]{}\frac{1}{2}$

Limiting values along two different paths ~~are~~ are different $0\neq\frac{1}{2}$, so the limit doesn't exist

(b) (5 points) For which of the integer values of k ($k=1,2,3,\ldots$), does the following limit exist?

$$\lim_{(x,y)\to(0,0)} \frac{xy^k}{x^2+y^6}$$

(Not very useful hint: though it's not the only way to go (and not necessarily the best), but in case you need it, you may use the inequality $ab\leq\frac{a^2+b^2}{2}$; it's possible to avoid using this)

If $k=1$: $\lim \frac{xy}{x^2+y^6}$ doesn't exist : take path $x=y^3$: $f=\frac{y^4}{y^6+y^6}\to\infty$ (as $y\to 0$)

If $k=2$: $\lim \frac{xy^2}{x^2+y^6}$ doesn't exist : take path $x=y^3$: $f=\frac{y^5}{y^6+y^6}\longrightarrow\infty$ (as $y\to 0$)

If $k=3$: $\lim \frac{xy^3}{x^2+y^6}$ doesn't exist : shown in (a)

If $k\geq 4$ limit exists :

Method 1 : let's make substitution $y^3=v$, i.e. $y=v^{1/3}$:

$\lim\limits_{(x,v)\to(0,0)} \frac{x\,v^{k/3}}{x^2+v^2}$. Now let's use polar coordinates $x=r\cos\theta$, $v=r\sin\theta$:

$$\lim_{\substack{r\to 0\\ \text{unif. in }\theta}} \frac{r\cos\theta\, r^{k/3}\sin^{k/3}\theta}{r^2}=\lim_{\substack{r\to 0\\ \text{unif. in }\theta}} r^{\frac{k}{3}-1}\cos\theta\sin^{k/3}\theta$$

If $k\geq 4$ then $\frac{k}{3}-1>0$, so $\lim\limits_{r\to 0} r^{\frac{k}{3}-1}=0$, and Squeeze theorem works:

Extra scratch paper

$$0 \le \left| r^{\frac{k}{3}-1} \cos\theta \sin^{k/3}\theta \right| \le r^{k/3}$$

$$\downarrow \qquad\qquad \downarrow$$

$$0 \qquad\qquad 0$$

So $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^k}{x^2+y^6} = \lim\limits_{(x,v)\to(0,0)} \dfrac{x\,v^{k/3}}{x^2+v^2} = \lim\limits_{\substack{r\to 0\\ \text{unif. in } \theta}} r^{\frac{k}{3}-1} \cos\theta \sin^{k/3}\theta = 0$

ANSWER: Limit exists if $k \ge 4$ only

Method 2: by the given inequality $ab \le \dfrac{a^2+b^2}{2}$, so

$$xy^3 \le \frac{x^2+y^6}{2}, \text{ so}$$

$$\frac{1}{x^2+y^6} \le \frac{1}{2|xy^3|}, \text{ so}$$

$$0 \le \left| \frac{xy^k}{x^2+y^6} \right| \le \frac{|xy^k|}{2|xy^3|} = \frac{1}{2}|y|^{k-3}$$

$\downarrow$ $\downarrow$ as $\begin{matrix} x\to 0 \\ y \to 0 \end{matrix}$ but only if $k-3>0$, i.e. $k \ge 4$

0 0

So by Squeeze thm $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^k}{x^2+y^6} = 0$ if $k \ge 4$

For $k \le 3$ the arguments should be as before.