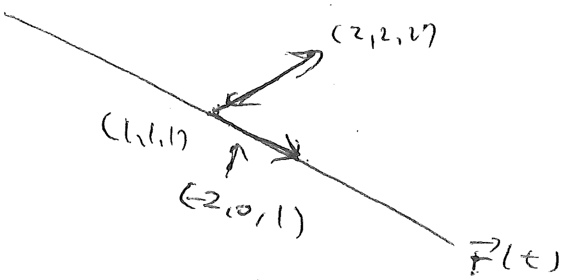
Problem 1

Parts (a) and (b) are unrelated.

(a) (6 points) Find an equation of the plane that contains the line $\vec{r}(t) = \langle 1-2t, 1, 1+t\rangle$ and contains the point $(2,2,2)$.

We start by choosing two vectors on the plane. To do so, we find one point on the plane other than (2,2,2). Let t=0. Then $\vec{r}(1) = \langle 1,1,1\rangle$. Thus, $\langle 2,2,2\rangle - \langle 1,1,1\rangle = \langle 1,1,1\rangle$ ~~$\langle 2,2,2\rangle - \langle 2,2,2\rangle = \langle +1,+1,+1\rangle$~~ ~~$\langle 1,1,1\rangle - \langle 2,2,2\rangle$~~ is on the plane, and $\langle -2,0,1\rangle$ as well (directional vec. of $\vec{r}(t)$).



Take the cross prod. to find the normal vector:

$$\langle 1,1,1\rangle \times \langle -2,0,1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -2 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix}$$

$$= \vec{i} - 3\vec{j} + 2\vec{k}.$$

Plug in (2,2,2) to get

Ans: $\boxed{x - 3y + 2z = 2 - 3(2) + 2(2) = 0}$

(b) (6 points) Find the intersection of the plane $x + y - z = 5$ and the line

$x = 1$
$y = 1 - 2t$
$z = 1 - 4t$

$x + y - z = 5$

$1 + (1-2t) - (1-4t)$

$= 1 + 1 - 2t - 1 + 4t = 2t + 1 = 5$

$\therefore t = 2.$

Ans) $\boxed{\therefore (x,y,z) = (1,-3,-7).}$

Problem 2

Parts (a) and (b) and (c) are unrelated.

(a) (2 points) Consider the parallelogram $ABCD$ (see the picture), and let $\vec{a} = \overrightarrow{AB}$ and $\vec{b} = \overrightarrow{AC}$. Express $\overrightarrow{AD} + \overrightarrow{BC}$ in terms of $\vec{a}$ and $\vec{b}$ only.

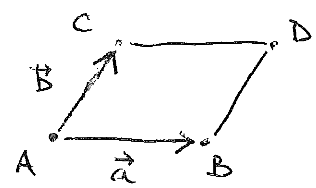
Note that $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \vec{a} + \vec{b}$

and

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\vec{a} + \vec{b}$

so

$\overrightarrow{AD} + \overrightarrow{BC} = (\vec{a} + \vec{b}) + (-\vec{a} + \vec{b}) = 2\vec{b}$



(b) (2 points) State the Triangle Inequality for vectors.

For any two vectors $\vec{v}$ and $\vec{u}$:

$\|\vec{v} + \vec{u}\| \leq \|\vec{v}\| + \|\vec{u}\|$

(c) (2+4 points) Describe **and** find the equation of the set of all points $P(x, y, z)$ that satisfy the property that the angle between $\overrightarrow{OP}$ and $\langle 0, 0, 1\rangle$ is equal to the angle between $\overrightarrow{OP}$ and $\langle 1, 2, 0\rangle$. Find the equation of this set. (Note: as usual, here O is the origin point $(0, 0, 0)$)

Angle between $\langle 0,0,1\rangle$ and $\overrightarrow{OP}$: $\cos^{-1} \dfrac{\overrightarrow{OP} \cdot \langle 0,0,1\rangle}{\|\overrightarrow{OP}\| \cdot \|\langle 0,0,1\rangle\|} = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

Angle between $\langle 1,2,0\rangle$ and $\overrightarrow{OP}$: $\cos^{-1} \dfrac{\overrightarrow{OP} \cdot \langle 1,2,0\rangle}{\|\overrightarrow{OP}\| \cdot \|\langle 1,2,0\rangle\|} = \dfrac{x+2y}{\sqrt{x^2+y^2+z^2} \cdot \sqrt{5}}$

So $\dfrac{z}{\sqrt{x^2+y^2+z^2}} = \dfrac{x+2y}{\sqrt{x^2+y^2+z^2} \cdot \sqrt{5}}$

$\Rightarrow \sqrt{5}\, z = x + 2y$

This is a plane

* Note: $(x,y,z) = (0,0,0)$ is one point on that plane that should actually be excluded from that plane.

Problem 3

Parts (a) and (b) are unrelated.

(a) (2 points) Finish the definition: a vector $\vec{v}$ is called a linear combination of vectors $\vec{u}$ and $\vec{w}$ if...

there exists $a, b \in \mathbb{R}$ (scalars) such that

$$\vec{v} = a\vec{u} + b\vec{w}$$

(b) (6 points) Let $\vec{a} = \langle -3, 1, -1 \rangle$ and $\vec{b} = \langle 1, 0, 2 \rangle$. Decompose vector $\vec{a}$ into the sum of two vectors, one of which is parallel to $\vec{b}$ and the other is perpendicular to $\vec{b}$.

$$\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$$

Method 1 :

$$\vec{a}_{\parallel} = \text{proj}_{\vec{b}}\,\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\right)\vec{b}$$

$$= \left(\frac{-5}{5}\right)\vec{b} = -\vec{b}$$

$$= \langle -1, 0, -2 \rangle$$

$$\vec{a}\cdot\vec{b} = -3 + 1\cdot 0 - 2 = -5$$

$$\vec{b}\cdot\vec{b} = 1^2 + 0^2 + 2^2 = 5$$

$$\vec{a}_{\perp} = \cancel{\text{proj}}\ \vec{a} - \vec{a}_{\parallel} = \langle -3, 1, -1 \rangle - \langle -1, 0, -2 \rangle$$

$$= \langle -2, 1, 1 \rangle$$

Method 2 :

Write $\vec{a} = \vec{u} + \vec{v}$ with $\vec{u} \parallel \vec{b}$ and $\vec{v} \perp \vec{b}$

Then $\vec{u} = t\vec{b} = \langle t, 0, 2t \rangle$ for some $t \in \mathbb{R}$

$$\vec{v} = \vec{a} - \vec{u} = \langle -3-t, 1, -1-2t \rangle$$

$$\vec{v} \perp \vec{b} \Rightarrow \vec{v}\cdot\vec{b} = 0 \Rightarrow (-3-t)1 + 0\cdot(-1) + 2(-1-2t) = 0$$

$$\Rightarrow 5t = -5 \Rightarrow t = -1$$

So $\vec{u} = \langle -1, 0, -2 \rangle$

$\vec{v} = \langle -2, 1, 1 \rangle$

Problem 4

Parts (a) and (b) are unrelated.

(a) (6 points) Suppose a particle travels in $\mathbb{R}^3$ with constant acceleration $\vec{r}''(t) = \langle 0, 2, -2\rangle$. Suppose the particle's initial (at $t = 0$) velocity is $\vec{r}'(0) = \langle 1, -2, 1\rangle$, and its position at time $t = 1$ is $\vec{r}(1) = \langle 6, 4, 5\rangle$. Find $\vec{r}(t)$ for any t.

$$\vec{r}'(t) = \int \vec{r}''(t)\,dt = \int \langle 0,2,-2\rangle\,dt = \langle 0, 2t, -2t\rangle + \langle c_1, c_2, c_3\rangle$$

$$\vec{r}'(0) = \langle 1,-2,1\rangle \Rightarrow \langle c_1, c_2, c_3\rangle = \langle 1,-2,1\rangle$$
$$\Rightarrow \vec{r}'(t) = \langle 1, 2t-2, -2t+1\rangle$$

$$\vec{r}(t) = \int \vec{r}'(t)\,dt = \int \langle 1, 2t-2, -2t+1\rangle\,dt$$
$$= \langle t, t^2-2t, -t^2+t\rangle + \langle d_1, d_2, d_3\rangle$$

$$\vec{r}(1) = \langle 6,4,5\rangle \Rightarrow \langle 1,-1,0\rangle + \langle d_1,d_2,d_3\rangle = \langle 6,4,5\rangle$$
$$\Rightarrow \langle d_1, d_2, d_3\rangle = \langle 5,5,5\rangle$$

$$\Rightarrow \boxed{\vec{r}(t) = \langle t+5, t^2-2t+5, -t^2+t+5\rangle}$$

(b) (6 points) Prove that for any vector function $\vec{w}(t)$ the following is true:

$$\frac{d}{dt}\Big(\vec{w}(t)\cdot(\vec{w}\,'(t)\times\vec{w}\,''(t))\Big) = \vec{w}(t)\cdot(\vec{w}\,'(t)\times\vec{w}\,'''(t))$$

Explain your steps

(Note: you may use any of the differentiation rules that were mentioned in class/in textbook).

$$\frac{d}{dt}\big(\vec{w}(t)\cdot(w'(t)\times w''(t))\big) = w'(t)\cdot(w'(t)\times w''(t)) + w(t)\cdot\Big(\frac{d}{dt}(w'(t)\times w''(t))\Big)$$

$$= \underbrace{w'(t)\cdot(w'(t)\times w''(t))}_{=0 \text{ b/c } w'(t)\times w''(t) \text{ is perp. to } w'(t)} + w(t)\cdot\Big(\underbrace{(w''(t)\times w''(t))}_{=0 \text{ b/c } w'' \text{ and } w'' \text{ are parallel}} + (w'(t)\times w'''(t))\Big)$$

$$= w(t)\cdot(w'(t)\times w'''(t)).$$